RF Basics; Contents

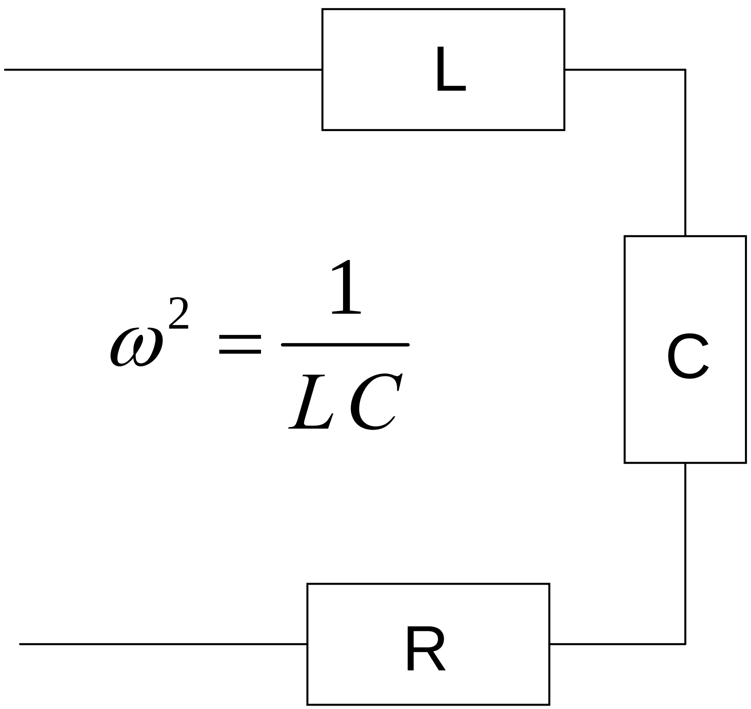
- Maxwell’s Equation
- Plane Wave
- Boundary Condition
- Cavity & RF Parameters
- Normal Mode Analysis
- Perturbation Theory
- Equivalent Circuit
- Coupled Cavity

Part-3

Contents of RF-3

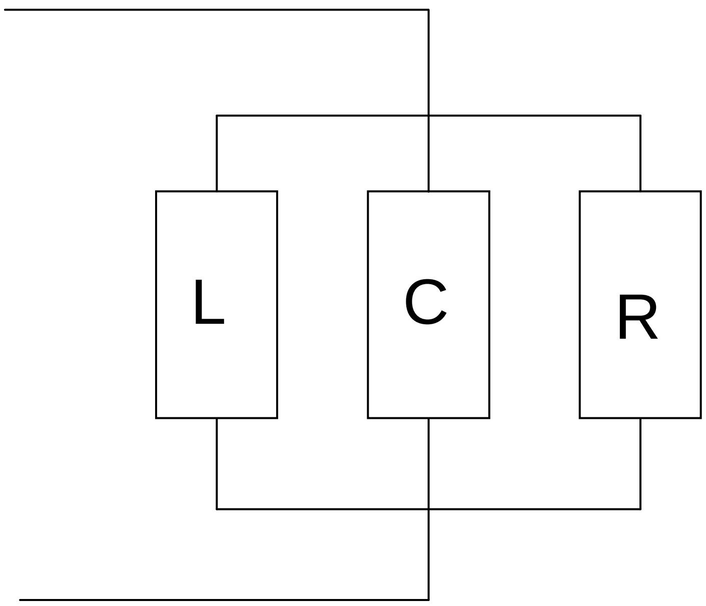
- Equivalent Circuit
- Coupled Cavity
- Excitation
- Cavity Measurements

Equivalent Circuit for Cavity



$$\omega^2 = \frac{1}{LC}$$

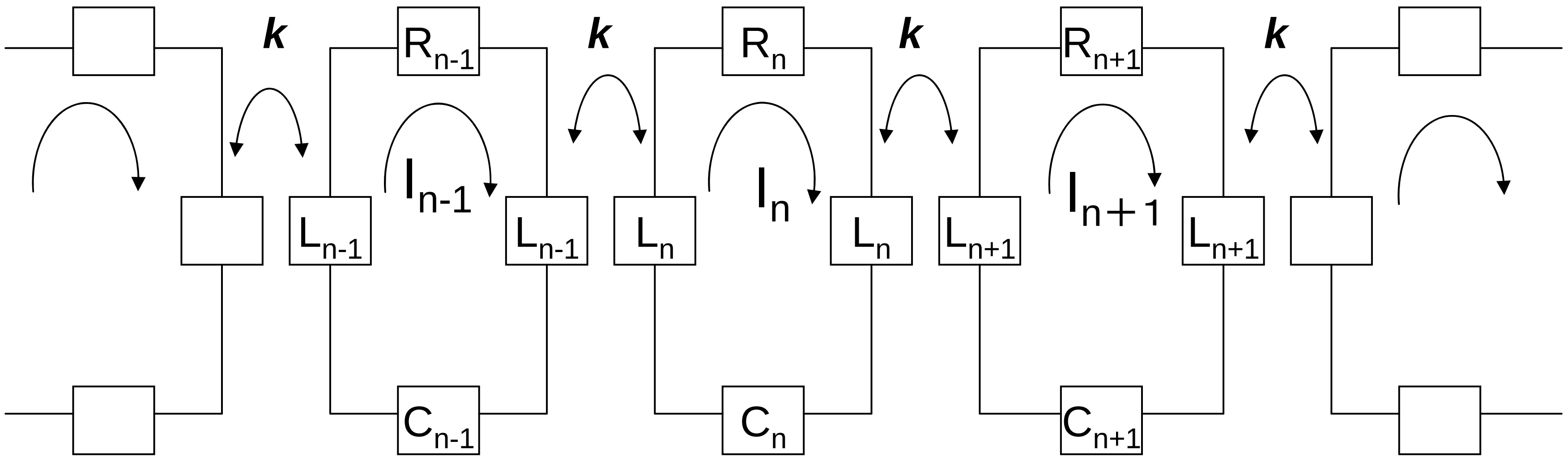
Series Circuit



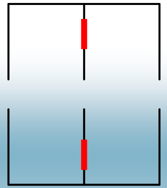
Parallel Circuit

Multi-cell Cavity / Coupled Resonator

Magnetic Coupling through Slot

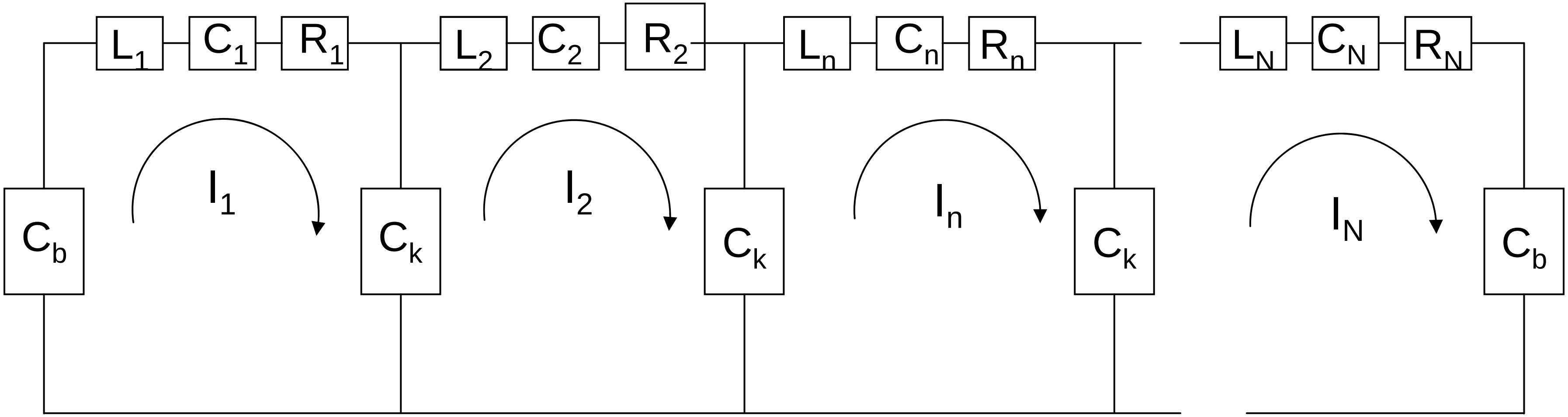


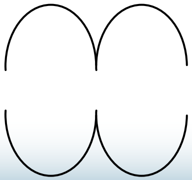
Cell #n



Coupled Resonator

Electric Coupling through Iris





Circuit Equation

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{C}I=0,\quad \text{Kirchhoff 's Rule and } I\propto\exp(j\omega t)$$

$$\frac{1}{C_b}I_1+\left(-\omega^2L_1+j\omega R_1+\frac{1}{C_1}\right)I_1+\frac{1}{C_k}(I_1-I_2)=0$$

$$\frac{1}{C_k}(I_n-I_{n-1})+\left(-\omega^2L_n+j\omega R_n+\frac{1}{C_n}\right)I_n+\frac{1}{C_k}(I_n-I_{n+1})=0$$

$$\frac{1}{C_k}(I_N-I_{N-1})+\left(-\omega^2L_N+j\omega R_N+\frac{1}{C_N}\right)I_N+\frac{1}{C_b}I_N=0$$

$$\left(-\omega^2 L_1 + j\omega R_1 + \frac{1}{C_1} + \frac{1}{C_b} + \frac{1}{C_k}\right) I_1 - \frac{1}{C_k} I_2 = 0$$

$$-\frac{1}{C_k} I_{n-1} + \left(-\omega^2 L_n + j\omega R_n + \frac{1}{C_n} + \frac{2}{C_k}\right) I_n - \frac{1}{C_k} I_{n+1} = 0$$

$$-\frac{1}{C_k} I_{N-1} + \left(-\omega^2 L_N + j\omega R_N + \frac{1}{C_N} + \frac{1}{C_b} + \frac{1}{C_k}\right) I_N = 0$$

$$\left(-\omega^2 L_1 + j\omega R_1 + \frac{1}{C_1'} \right) I_1 - \frac{1}{C_k} I_2 = 0, \quad \frac{1}{C_1'} = \frac{1}{C_1} + \frac{1}{C_b} + \frac{1}{C_k}$$

$$-\frac{1}{C_k} I_{n-1} + \left(-\omega^2 L_n + j\omega R_n + \frac{1}{C_n'} \right) I_n - \frac{1}{C_k} I_{n+1} = 0, \quad \frac{1}{C_n'} = \frac{1}{C_n} + \frac{2}{C_k}$$

$$-\frac{1}{C_k} I_{N-1} + \left(-\omega^2 L_N + j\omega R_N + \frac{1}{C_N'} \right) I_N = 0, \quad \frac{1}{C_N'} = \frac{1}{C_N} + \frac{1}{C_b} + \frac{1}{C_k}$$

Introduce Coupling Constant

$$\left(-\omega^2 L_1 C_1' + j\omega R_1 C_1' + 1\right) I_1 - k_1 I_2 = 0, \quad k_1 = \frac{C_1'}{C_k}$$

$$-k I_{n-1} + \left(-\omega^2 L_n C_n' + j\omega R_n C_n' + 1\right) I_n - k I_{n+1} = 0, \quad \boxed{k = \frac{C_n'}{C_k}}$$

$$-k_N I_{N-1} + \left(-\omega^2 L_N C_N' + j\omega R_N C_N' + 1\right) I_N = 0, \quad k_N = \frac{C_N'}{C_k}$$

$$-\omega^2 L_n C_n' + j\omega R_n C_n' = -\frac{\omega^2}{\omega_n^2}, \quad \frac{L_n C_n'}{\omega_n^2} = 1 - j\frac{R_n}{\omega L_n} = 1 - j\frac{1}{Q}$$

Field & Cell Frequency Relation

$$\left(1-\frac{\omega^2}{\omega_1^2}\right)I_1 - k_1 I_2 = 0 \qquad \frac{\omega_1}{\omega} = 1 \Big/ \sqrt{1-k_1\frac{I_2}{I_1}}$$

$$-k\,I_{n-1} + \left(1-\frac{\omega^2}{\omega_n^2}\right)I_n - k\,I_{n+1} = 0 \qquad \frac{\omega_n}{\omega} = 1 \Big/ \sqrt{1-k\frac{I_{n-1}+I_{n+1}}{I_n}}$$

$$-k_N\,I_{N-1} + \left(1-\frac{\omega^2}{\omega_N^2}\right)I_N = 0 \qquad \frac{\omega_N}{\omega} = 1 \Big/ \sqrt{1-k_N\frac{I_{N-1}}{I_N}}$$

Flat π-Mode

$$I_n = -I_{n\pm1}$$

$$\frac{\omega_\pi}{\omega_1} = \sqrt{1 + k_1}$$

$$\frac{\omega_\pi}{\omega_n} = \sqrt{1 + 2k}$$

$$\frac{\omega_\pi}{\omega_N} = \sqrt{1 + k_N}$$

0-Mode

$$I_n = I_{n\pm1}$$

$$\frac{\omega_0}{\omega_1} = \sqrt{1 - k_1}$$

$$\frac{\omega_0}{\omega_n} = \sqrt{1 - 2k}$$

$$\frac{\omega_0}{\omega_N} = \sqrt{1 - k_N}$$

$$\frac{\omega_\pi - \omega_0}{\omega_\pi + \omega_0} \cong k$$

Perturbation / Tuning

$$-k\,I_{n-1}+\left(1-\frac{\omega_\pi^2}{\omega_n^2}\right)I_n-k\,I_{n+1}=0\,,\qquad \frac{\omega_\pi^2}{\omega_n^2}=1+2k$$

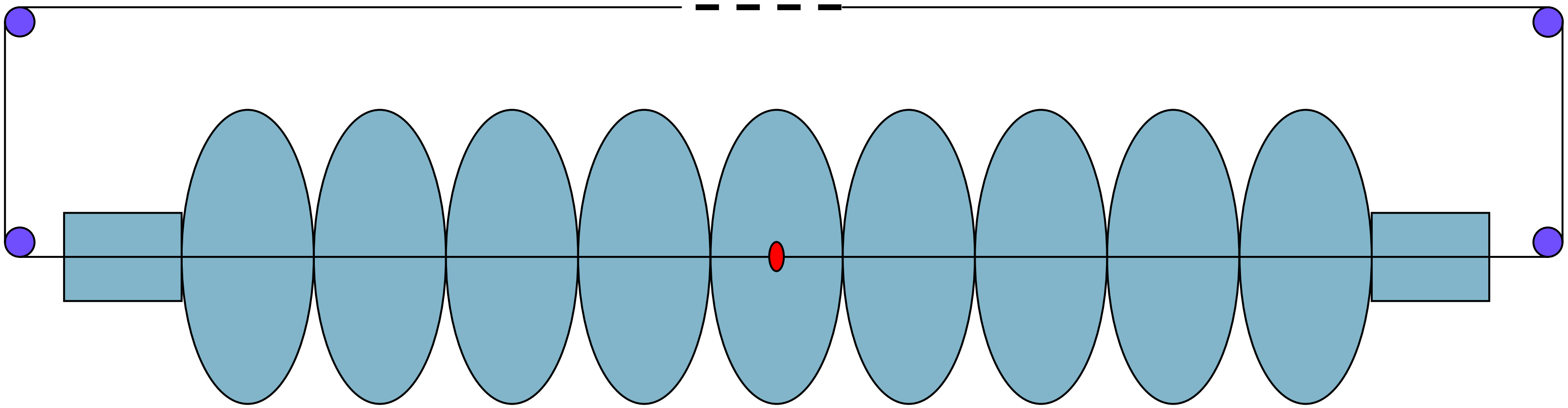
$$\omega_n\Rightarrow\omega_n+\delta\,\omega_n\,,\qquad I_n\Rightarrow I_n+\delta\,I_n$$

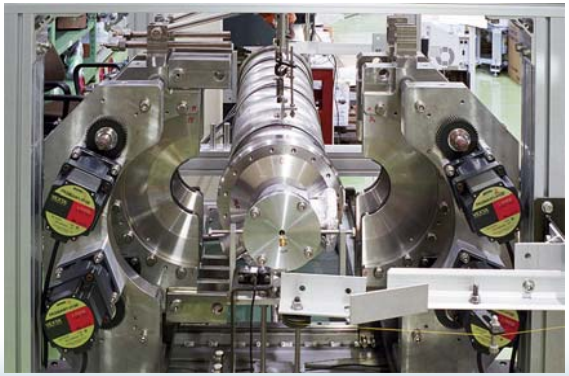
$$-k\,I_{n-1}+\left[1-\frac{\omega_\pi^2}{(\omega_n+\delta\omega_n)^2}\right](I_n+\delta I_n)-kI_{n+1}=0$$

$$-k\,I_{n-1}+\left[1-\frac{\omega_\pi^2}{\omega_n^2}\left(1-2\frac{\delta\,\omega_n}{\omega_n}\right)\right](I_n+\delta I_n)-kI_{n+1}=0$$

$$\frac{\delta\,I_n}{I_n}=\frac{1}{k}\frac{\delta\omega_n}{\omega_n}\,,\qquad \text{if } k\approx 2\%\ \&\ \delta I_n/I_n\approx 2\%\,,\ \delta\omega_n/\,\omega_n\approx 4\,\mathrm{x}10^{-4}$$

Beads pull Method





$$\frac{\Delta\omega}{\omega_0} \approx -\frac{X_{er}\left|\vec{E}\right|^2 + X_{mr}\left|\vec{H}\right|^2}{2W}\Delta V$$

Dispersion Relation

$$-k\,I_{n-1} + \left(1-\frac{\omega^2}{\omega_0^2}\right) I_n - k\,I_{n+1} = 0$$

$$I_n^0 = \sqrt{\frac{1}{N}}$$

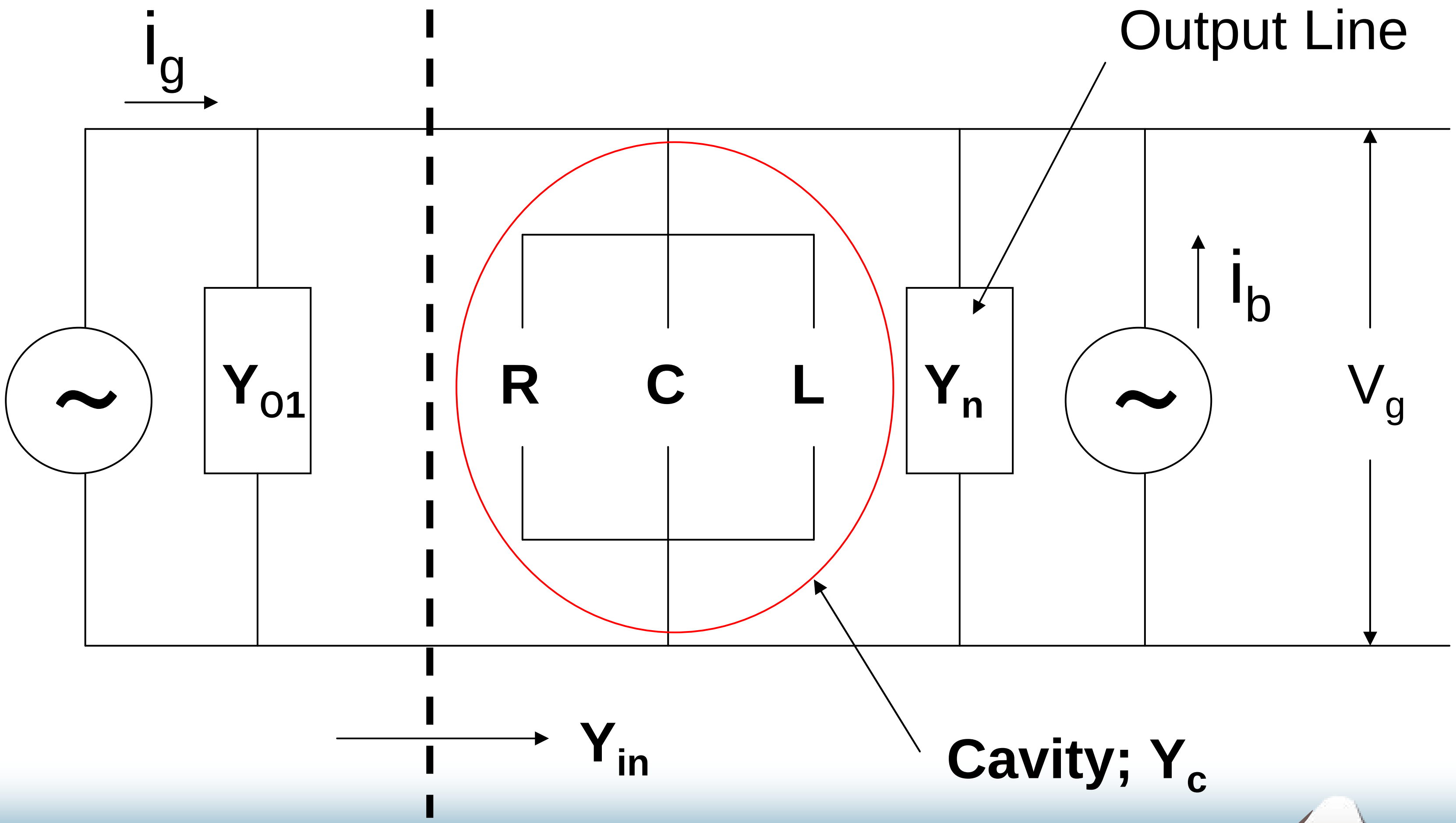
$$I_n^m = \sqrt{\frac{2}{N}}\cos\frac{mn\pi}{N}, \quad \frac{\omega_m^2}{\omega_0^2} = 1\Big/\left(1+2k\cos\frac{m\pi}{N}\right)$$

$$I_n^\pi = \sqrt{\frac{1}{N}}\cos(n\pi)$$

n ; cell #
m ; mode #
m=N ; π–Mode

Powering the Cavity

Equivalent Circuit



Available Power

$$P_g = \frac{i_g^2}{8\,Y_{01}}, \quad i_g = 2\sqrt{2\,P_g\,Y_{01}}$$

Cavity Voltage

$$V_g = \frac{i_g}{Y_{01} + Y_C + \sum_n Y_n} = \frac{2\sqrt{2\,P_g\,Y_{01}}}{Y_{01} + Y_C + \sum_n Y_n}$$

Cavity ; Parallel L,C,R Circuit

$$\frac{d^2}{dt^2}V + \frac{1}{RC}\frac{d}{dt}V + \frac{1}{LC}V = 0$$

$$\omega_0^2 = \frac{1}{LC}, \quad Q_0 = R\sqrt{\frac{C}{L}} = RC\omega_0$$

$$\frac{d^2}{dt^2}V + \frac{\omega_0}{Q_0}\frac{d}{dt}V + \omega_0^2 V = 0$$

$$Y_C = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right) = \frac{1}{R} + j\frac{Q_0}{R}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

$$\approx \frac{1}{R} + 2j\frac{Q_0}{R}\left(\frac{\omega - \omega_0}{\omega_0}\right)$$

$$Y_{01} + Y_C + \sum_n Y_n = \frac{Q_0}{R}\left(\frac{R}{Q_0}Y_{01} + \sum_n \frac{R}{Q_0}Y_n + \frac{1}{Q_0}\right) + 2j\frac{Q_0}{R}\left(\frac{\omega - \omega_0}{\omega_0}\right)$$

$$Q_{in} = \frac{Q_0}{RY_{01}}, \quad Q_n = \frac{Q_0}{RY_n}, \quad \frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_{in}} + \sum_n \frac{1}{Q_n}$$

$$Y_{01} + Y_C + \sum_n Y_n = \frac{Q_0}{R\,Q_L} + 2j\frac{Q_0}{R}\left(\frac{\omega - \omega_0}{\omega_0}\right) = \frac{Q_0}{R\,Q_L}\left(1 - j\tan\psi\right)$$

$$\tan\psi \equiv -2Q_L\frac{\omega - \omega_0}{\omega_0}, \quad \psi\ ;\ \text{Detuning Angle}$$

$$V_g = \frac{i_g}{Y_{01} + Y_C + \sum_n Y_n} = \frac{2\sqrt{2P_g Y_{01}}\,\dfrac{R\,Q_L}{Q_0}}{1 - j\tan\psi}$$

$$= 2\sqrt{2P_g\frac{R\,Q_L^2}{Q_0\,Q_{in}}}\cos\psi\exp\left(j\psi\right) = 2\sqrt{P_g\left(\frac{R}{Q}\right)\frac{Q_L^2}{Q_{in}}}\cos\psi\exp\left(j\psi\right)$$

Transmission (T_n) / Output

$$i_n = \frac{Y_n\, i_g}{Y_{01} + \sum_n Y_n + \frac{1}{R} + 2j\frac{Q_0}{R}\left(\frac{\omega - \omega_0}{\omega_0}\right)}, \quad P_n = \frac{\left|i_n\right|^2}{2Y_n}$$

$$T_n \equiv \frac{P_n}{P_g} = \frac{4Y_{01}\,Y_n}{\left(Y_{01} + \sum_n Y_n + \frac{1}{R}\right)^2 + 4\left(\frac{Q_0}{R}\right)^2\left(\frac{\omega - \omega_0}{\omega_0}\right)^2}$$

$$= \frac{4\frac{R}{Q_0}Y_{01}\frac{R}{Q_0}Y_n}{\left(\frac{R}{Q_o}Y_{01} + \frac{R}{Q_0}\sum_n Y_n + \frac{1}{Q_0}\right)^2 + 4\left(\frac{\omega - \omega_0}{\omega_0}\right)^2}$$

Reflection

$$\Gamma = \frac{1 - Y_{in}/Y_{01}}{1 + Y_{in}/Y_{01}}$$

$$Y_{in} = \frac{1}{R} + 2j\frac{Q_0}{R}\left(\frac{\omega - \omega_0}{\omega_0}\right) + \sum_n Y_n$$

$$|\Gamma| = \sqrt{\frac{P_{ref}}{P_g}} = \left|\frac{1-\beta^*}{1+\beta^*}\right|, \quad \text{at Resonance}$$

$$\beta^* = \frac{1+|\Gamma|}{1-|\Gamma|}, \quad \text{Over Coupling}$$

$$\beta^* = \frac{1-|\Gamma|}{1+|\Gamma|}, \quad \text{Under Coupling}$$

$$\beta_{in} = \frac{Q_0}{Q_{in}}, \quad \beta_n = \frac{Q_0}{Q_n}$$

$$\frac{Y_{in}}{Y_{01}} = \frac{1 + \sum_n \beta_n}{\beta_{in}} + 2j\left(\frac{\omega - \omega_0}{\omega_0}\right) Q_{in}$$

$$\beta^* \equiv \frac{1 + \sum_n \beta_n}{\beta_{in}}$$

$$T_n = \frac{P_n}{P_g} = \frac{4\dfrac{1}{Q_{in}}\dfrac{1}{Q_n}}{\left(\dfrac{1}{Q_{in}} + \dfrac{1}{Q_0} + \sum_n \dfrac{1}{Q_n}\right)^2 + 4\left(\dfrac{\omega - \omega_0}{\omega_0}\right)^2},$$

$$= \frac{4Q_L^2}{Q_{in}\,Q_n}\frac{1}{1 + 4Q_L^2\left(\dfrac{\omega - \omega_0}{\omega_0}\right)^2}$$

Band Width ; $T_n = 0.5$

Cavity ; Parallel L,C,R Circuit

$$\frac{d^2}{dt^2}V + \frac{1}{RC}\frac{d}{dt}V + \frac{1}{LC}V = 0$$

$$\omega_0^2 = \frac{1}{LC}, \quad Q_0 = R\sqrt{\frac{C}{L}} = RC\omega_0$$

$$\frac{d^2}{dt^2}V + \frac{\omega_0}{Q_0}\frac{d}{dt}V + \omega_0^2 V = 0$$

Cavity Voltage Equation

$$\frac{d^2}{dt^2}V + \left(1 - \frac{1-j}{Q_0}\right)\omega_a^2 V = 0, \quad \omega \cong \omega_a\left(1 - \frac{1-j}{2Q_0}\right)$$ E

$$\frac{d^2}{dt^2}V + (1+j)\frac{\omega_a}{Q_0}\frac{d}{dt}V + \omega_a^2 V = 0, \quad \omega \cong \omega_a\left(1 - \frac{1-j}{2Q_0}\right)$$ H

$$\frac{d^2}{dt^2}V + \frac{\omega_a}{Q_0}\frac{d}{dt}V + \omega_a^2 V = 0, \quad \omega \cong \omega_a\left(1 + \frac{j}{2Q_0}\right)$$

Cavity Voltage Equation

$$\frac{d^2}{dt^2}\vec{V}(t) + \left(\frac{1}{Q_L} + j\frac{1}{Q_o}\right)\omega_0 \frac{d}{dt}\vec{V}(t) + \omega_o^2 \vec{V}(t) = U(t)$$

Loss

Skin Depth

Drive Force

$\propto \exp(j\omega t)$

Cavity Voltage

During Build-up , Step Pulse Response

$$\vec{V} = V_d \left[1 - \exp\left(-\frac{t}{T_F} \right) \exp\left(j \frac{\tan\psi}{T_F} t \right) \right] \cos\psi \, \exp\{ j(\theta + \psi) \}$$

$$V_d = V_g = 2\sqrt{P_g \left(\frac{R}{Q} \right) Q_0 \frac{\beta}{(1+\beta)^2}}$$

$$\vec{V} = \vec{V}_{FlatTop} \quad \text{at Beam Timing } T_e = T_F \ \mathrm{In} \frac{1+\beta+\beta_b}{\beta_b}$$

$$\vec{V} = \tilde{V} \exp\left(j\omega t \right)$$

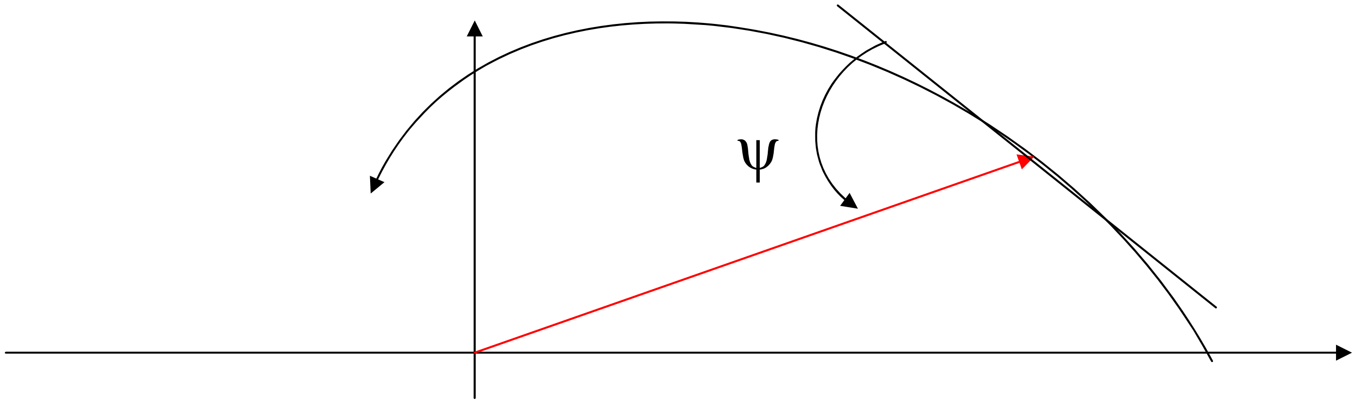
$$\tilde{V} = \tilde{V}_d + (\tilde{V}_o - \tilde{V}_d)\exp(-\frac{t}{T_F})\exp(-j\frac{\tan\psi}{T_F}t)$$

$$T_F \equiv \frac{2Q_L}{\omega_0}; \quad \text{Filling Time}$$

$$\tan\psi \equiv -2Q_L\left(\frac{\omega-\omega_0}{\omega_0}\right); \quad \text{Detuning Angle}\left(\psi\right)$$

Equi-angular Spiral

$$\tilde{V} = \tilde{V}_o \exp\left(-\frac{t}{T_F}\right) \exp\left(j\frac{\tan\psi}{T_F}t\right)$$



Cavity Voltage

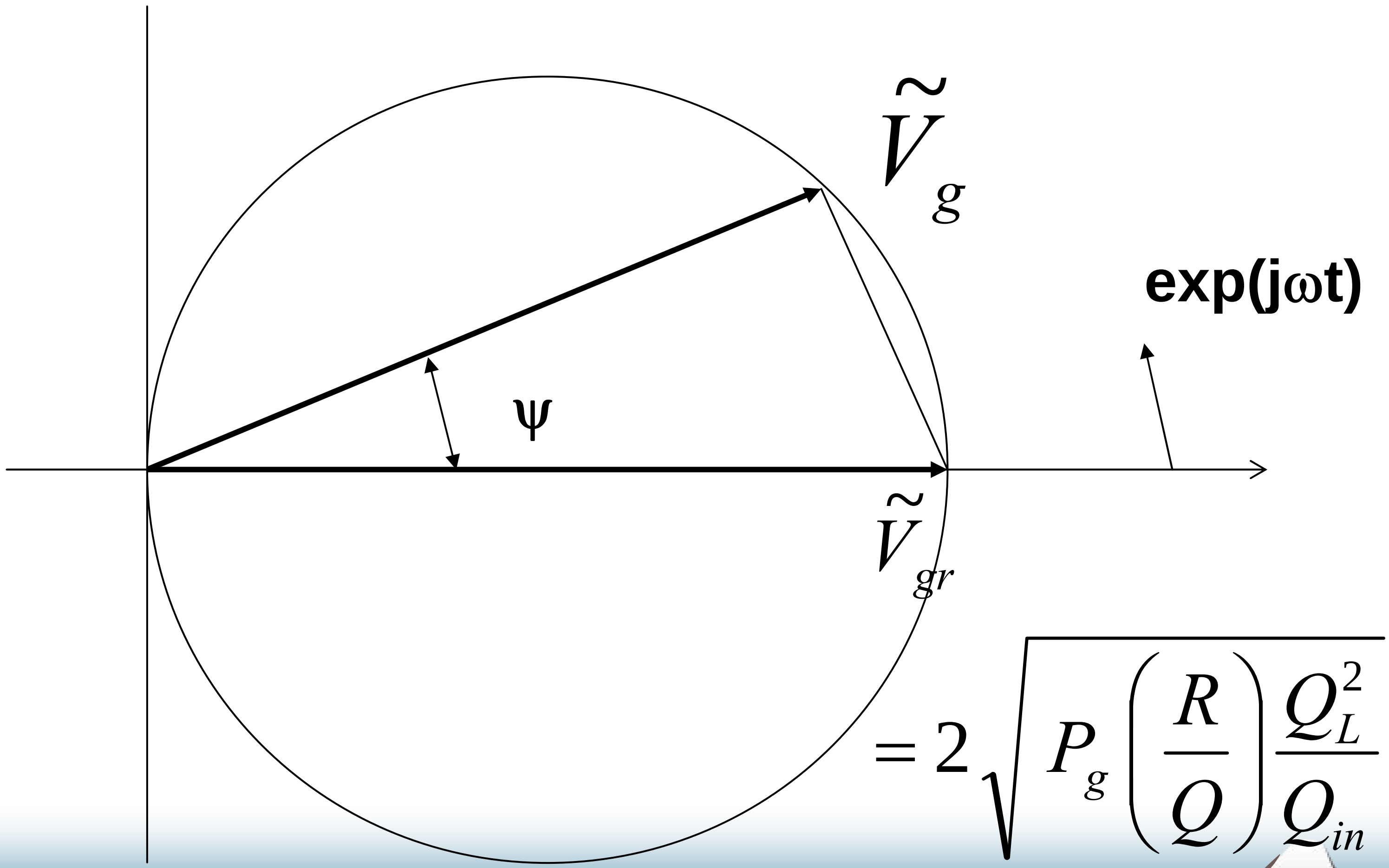
at the CW Limit with Beam

$$\tilde{V} = \left[2\sqrt{P_g \left(\frac{R}{Q}\right) Q_o \frac{\beta}{(1+\beta)^2}} \exp(j\theta) - I_b \left(\frac{R}{Q}\right) Q_o \frac{1}{1+\beta} \right] \cos\psi \exp(j\psi)$$

Beam Voltage

Feedback works fine.

Phaser Diagram



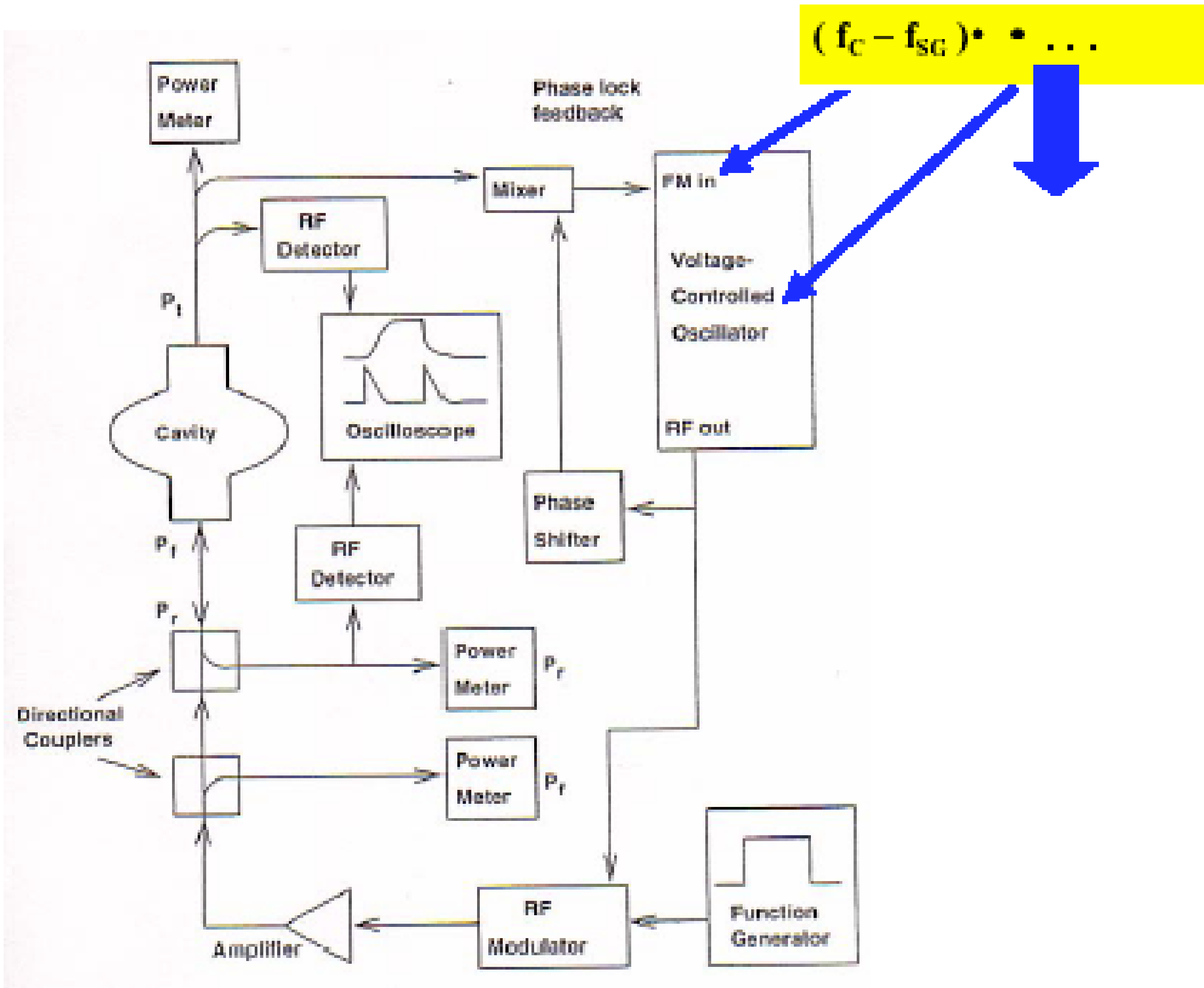
$$= 2\sqrt{P_g \left(\frac{R}{Q}\right) \frac{Q_L^2}{Q_{in}}}$$

Cavity Measurement

$$\text{Decay}\,;W \propto \exp\left(-\frac{\omega_0}{Q_L}t\right), \quad \text{Band Width}\,;Q_L = \frac{\omega_0}{2\delta\omega} \Rightarrow Q_L$$

$$P_g, P_r, P_n \Rightarrow P_o, \beta^*, \beta_n, \beta_{in}, Q_o, Q_{in}, Q_n$$

$$E_{acc} = \frac{1}{L_{Cavity}}\sqrt{\left(\frac{R}{Q}\right)\omega W} = \frac{1}{L_{Cavity}}\sqrt{\left(\frac{R}{Q}\right)P_n Q_n}$$


$(f_C - f_{SG})\bullet$ • . . .
Power Meter
Phase lock feedback
Mixer
FM in
RF Detector
Voltage-Controlled Oscillator
P_t
Cavity
Oscilloscope
RF out
Phase Shifter
P_f
RF Detector
P_r
Power Meter
P_r
Directional Couplers
Power Meter
P_f
Function Generator
RF Modulator
Amplifier

Normal Mode Analysis（Slator)

- Igen Value ; Frequency
- Igen Function ; Field Distribution $\vec{E}_a \, , \vec{H}_a$

$$\int_V \vec{E}_a \cdot \vec{E}_b \, dV = \delta_{ab} \quad , \quad \int_V \vec{H}_a \cdot \vec{H}_b \, dV = \delta_{ab}$$

$$\vec{E} = \sum_a \mathrm{e}_a \vec{E}_a = \sum_a \vec{E}_a \int_V \vec{E}_a \cdot \vec{E} \, dV$$

$$\vec{H} = \sum_a \mathrm{h}_a \vec{H}_a = \sum_a \vec{H}_a \int_V \vec{H}_a \cdot \vec{H} \, dV$$

Time Dependence

$$\varepsilon\mu\frac{d^2}{dt^2}\int_V \vec{E}\cdot\vec{E}_a\,dV + k_a^2\int_V \vec{E}\cdot\vec{E}_a\,dV$$

$$= -\mu\frac{d}{dt}\left[\int_V \vec{J}\cdot\vec{E}_a\,dV - \int_{S'}\left(\vec{n}\times\vec{H}\right)\cdot\vec{E}_a\,dS\right] - k_a\int_S\left(\vec{n}\times\vec{E}\right)\cdot\vec{H}_a\,dS$$

Dielectric Loss **External Drive** **Wall Loss**

$$\varepsilon\mu\frac{d^2}{dt^2}\int_V \vec{H}\cdot\vec{H}_a\,dV + k_a^2\int_V \vec{H}\cdot\vec{H}_a\,dV$$

$$= k_a\left[\int_V \vec{J}\cdot\vec{E}_a\,dV - \int_{S'}\left(\vec{n}\times\vec{H}\right)\cdot\vec{E}_a\,dS\right] - \varepsilon\frac{d}{dt}\int_S\left(\vec{n}\times\vec{E}\right)\cdot\vec{H}_a\,dS$$

No Dielectric

$$\varepsilon\mu \frac{d^2}{dt^2} \mathrm{e}_a + k_a^2 \mathrm{e}_a$$

$$= \mu \frac{d}{dt}\left[\int_{S'} \left(\vec{n} \times \vec{H}\right) \cdot \vec{E}_a \, dS\right] - k_a \int_S \left(\vec{n} \times \vec{E}\right) \cdot \vec{H}_a \, dS$$

$$\frac{d^2}{dt^2} \mathrm{e}_a + \omega_a^2 \mathrm{e}_a$$

$$= \frac{1}{\varepsilon} \frac{d}{dt}\left[\int_{S'} \left(\vec{n} \times \vec{H}\right) \cdot \vec{E}_a \, dS\right] - \frac{\omega_a}{\sqrt{\varepsilon\mu}} \int_S \left(\vec{n} \times \vec{E}\right) \cdot \vec{H}_a \, dS$$

Wall Loss & Q

RF Surface Registance ; $R_S = \sqrt{\dfrac{\omega\mu}{2\sigma_{\text{wall}}}} = \dfrac{1}{\delta\,\sigma_{\text{wall}}}$

Wall Loss ; $P_0 = \dfrac{R_S}{2}\int_S \left|\vec{H}\right|^2 dS$

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma_{\text{wall}}}}$$

$$\vec{n}\times\vec{E} = (1+j)\sqrt{\frac{\omega\mu}{2\sigma_{\text{wall}}}}\,\vec{H} = (1+j)\,R_S\,\vec{H}$$

$$\int_V \vec{H}\cdot\vec{H}_a\,dV = j\sqrt{\frac{\varepsilon}{\mu}}\int_V \vec{E}\cdot\vec{E}_a\,dV$$

$$\frac{d^2}{dt^2}\mathrm{e}_a + \omega_a^2 \mathrm{e}_a$$

$$= \frac{1}{\varepsilon}\frac{d}{dt}\left[\int_{S'}\left(\vec{n}\times\vec{H}\right)\cdot\vec{E}_a\, dS\right] - \frac{\omega_a}{\sqrt{\varepsilon\mu}}\int_S\left(\vec{n}\times\vec{E}\right)\cdot\vec{H}_a\, dS$$

$$= \frac{1}{\varepsilon}\frac{d}{dt}\left[\int_{S'}\left(\vec{n}\times\vec{H}\right)\cdot\vec{E}_a\, dS\right] + (1-j)\,\omega_a\,\mathrm{e}_a\,\frac{R_S}{\mu}\int_s\left|\vec{H}_a\right|^2 dS$$

$$Q_0 = \frac{\omega_a W}{P_0}, \quad W = \frac{\mu}{2} \quad \Longrightarrow \quad (1-j)\frac{\omega_a^2}{Q_0}\mathrm{e}_a$$

No External Drive

$$\frac{d^2}{dt^2}\mathrm{e}_a + \left(1 - \frac{1-j}{Q_0}\right)\omega_a^2\,\mathrm{e}_a = 0$$

$$\mathrm{e}_a \propto \exp(j\omega t),\quad \omega = \omega_1 + j\omega_2$$

$$\omega_1 \cong \omega_a\left(1 - \frac{1}{2Q_0}\right),\quad \omega_2 \cong \frac{\omega_a}{2Q_0}$$

$$\mathrm{e}_a \propto \exp\left(-\frac{\omega_a}{2Q_0}t\right)\exp(j\omega_1 t)$$

General Solution

$$\vec{V} = \vec{A}\exp\left(j\omega_0 t\right)\exp\left(-\frac{\omega_0}{2Q_L}t\right) + \vec{V_d}\exp\left(j\omega t\right)$$

$$\vec{V} = 0; \quad \text{at } t = 0 \quad then \quad \vec{A} = -\vec{V_d}$$

$$\vec{V} = \vec{V_d}\left\{1 - \exp\left(-\frac{\omega_0}{2Q_L}t\right)\exp\left(-j\frac{\tan\psi}{T_F}t\right)\right\}\exp\left(j\omega t\right)$$