RF Basics; Contents

- Maxwell’s Equation
- Plane Wave
- Boundary Condition
- Cavity & RF Parameters
- Normal Mode Analysis
- Perturbation Theory
- Equivalent Circuit
- Coupled Cavity

RF-2

Summary of RF-1

- Maxwell → Wave Equation
- Particular & Practical Solution

  Plane Wave, TE, TM, TEM Modes
- Boundary Condition
- Reflection, Surface Impedance
- Resonant Cavity Modes

Contents of RF-2

- Normal Mode Analysis (Perfect Conductor)
- Application to Real Cavity
- Time Dependence of Field
- Perturbation
- Frequency Tuning
- Field Distribution Measurement

General Analysis on Resonant Cavity

Any Vector $\vec{A} = \vec{B} + \vec{C}\,,$

$\vec{B}\,;$ Solenoidal, $div\,\vec{B} = 0\,,\quad \vec{B} = rot\,\vec{B}'$

$\vec{C}\,;$ Irrotational, $rot\,\vec{C} = 0\,,\quad \vec{C} = grad\,\varphi$

$$\vec{E} \Rightarrow \vec{E} + \vec{F}\,,\quad \vec{H} \Rightarrow \vec{H} + \vec{0}$$

Solenoidal Components

$$rot\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Rightarrow \quad rot\,\vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$rot\,\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \Rightarrow \quad rot\,\vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

Irrotational Components

$$\vec{0} = \sigma \vec{F} + \varepsilon \frac{\partial \vec{F}}{\partial t}$$

$$div\,\vec{D} = \rho \quad \Rightarrow \quad div\,\vec{F} = \rho$$

Solenoidal Components

$$rot\,\vec{E} = -j\,\omega\,\mu\,\vec{H}\,; \quad rot\,\vec{H} = \left(\sigma + j\,\omega\,\varepsilon\right)\vec{E}$$

$$\nabla^2\,\vec{E} + k^2\,\vec{E} = 0\,; \quad \nabla^2\,\vec{H} + k^2\,\vec{H} = 0$$

$$k^2 = \omega^2\,\varepsilon\,\mu - j\,\omega\,\mu\,\sigma$$

To solve this Wave Equation
Clear Boundary Condition is necessary.

Perfect Conductor

$$k_a \vec{E}_a = rot\, \vec{H}_a \,, \quad k_a \vec{H}_a = rot\, \vec{E}_a$$

$$\vec{n} \,\mathrm{x}\, \vec{E}_a = 0 \;\; on\, S \quad \Rightarrow \quad \vec{n} \cdot \vec{H}_a = 0$$

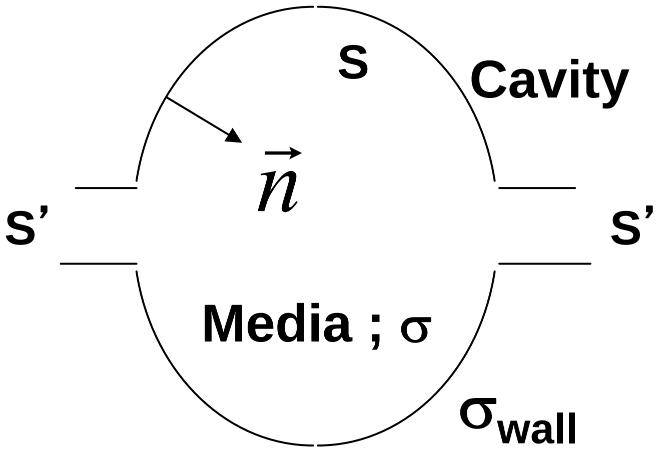
$$\vec{n} \,\mathrm{x}\, \vec{H}_a = 0 \;\; on\, S^{'} \quad \Rightarrow \quad \vec{n} \cdot \vec{E}_a = 0$$

$$\nabla^2 \vec{E}_a + k_a^2 \vec{E}_a = 0 \,, \quad \nabla^2 \vec{H}_a + k_a^2 \vec{H}_a = 0$$

$$k_a^2 = \omega_a^2 \varepsilon \mu - j \omega_a \mu \sigma$$

Can be solved Analytically or by Computer Codes

- Boundary Condition
- Short-Circuited Plane S
- Open-Circuited Plane S'



$$\vec{n} \times \vec{E}_a = 0\,,\ \vec{n} \bullet \vec{H}_a = 0 \ \text{ on S}$$; **Perfect Conductor**

$$\vec{n} \times \vec{H}_a = 0\,,\ \vec{n} \bullet \vec{E}_a = 0 \ \text{ on S'}$$

Useful Equation

$$\int_V div\,\vec{C}\,dV = \int_S \vec{n}\cdot\vec{C}\,dS\,; \quad \text{Gauss}$$

$$\vec{A}\cdot\left(\vec{B}\,\text{x}\,\vec{C}\right)=\vec{B}\cdot\left(\vec{C}\,\text{x}\,\vec{A}\right)=\vec{C}\cdot\left(\vec{A}\,\text{x}\,\vec{B}\right)$$

$$div\left(\vec{A}\,\text{x}\,\vec{B}\right)=\vec{B}\cdot rot\,\vec{A}-\vec{A}\cdot rot\,\vec{B}$$

$$\int_V div\left(\vec{A}\,\text{x}\,\vec{B}\right)dV = \int_S \vec{n}\cdot\left(\vec{A}\,\text{x}\,\vec{B}\right)dS$$

$$= \int_S \vec{B}\cdot\left(\vec{n}\,\text{x}\,\vec{A}\right)dS = -\int_S \vec{A}\cdot\left(\vec{n}\,\text{x}\,\vec{B}\right)dS$$

$$\vec{n}\,\text{x}\,\vec{E}_a = 0 \;\; on\; S\,, \quad \vec{n}\,\text{x}\,\vec{H}_a = 0 \;\; on\; S'$$

Orthogonality

$$k_a\, div\left(\vec{E}_b \times \vec{H}_a\right) - k_b\, div\left(\vec{E}_a \times \vec{H}_b\right)$$

$$= div\left(\vec{E}_b \times rot\, \vec{E}_a\right) - div\left(\vec{E}_a \times rot\, \vec{E}_b\right)$$

$$= rot\, \vec{E}_a \cdot rot\, \vec{E}_b - \vec{E}_b \cdot rot\, rot\, \vec{E}_a - rot\, \vec{E}_b \cdot rot\, \vec{E}_a - \vec{E}_a \cdot rot\, rot\, \vec{E}_b$$

$$= \left(k_b^2 - k_a^2\right)\vec{E}_a \cdot \vec{E}_b$$

$$\left(k_b^2 - k_a^2\right)\int_V \vec{E}_a \cdot \vec{E}_b\, dV = k_a \int_V div\left(\vec{E}_b \times \vec{H}_a\right) dV - k_b \int_V div\left(\vec{E}_a \times \vec{H}_b\right) dV$$

$$= k_a \int_{S,S'} \vec{n} \cdot \left(\vec{E}_b \times \vec{H}_a\right) dS - k_b \int_{S,S'} \vec{n} \cdot \left(\vec{E}_a \times \vec{H}_b\right) dS$$

$$= 0$$

$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \vec{B} \cdot \left(\vec{C} \times \vec{A}\right) = \vec{C} \cdot \left(\vec{A} \times \vec{B}\right)$$

$$\boxed{\vec{n} \times \vec{E}_{a,b} = 0 \text{ on } S, \quad \vec{n} \times \vec{H}_{a,b} = 0 \text{ on } S'}$$

Normalization

$$div\left(\vec{E}_a \,\mathrm{x}\, rot\,\vec{E}_a\right) = \left(rot\,\vec{E}_a\right)^2 - \vec{E}_a \cdot rot\,rot\,\vec{E}_a = k_a^2\left(\vec{H}_a^2 - \vec{E}_a^2\right)$$

$$\int_{S,S'} \vec{n}\cdot\left(\vec{E}_a \,\mathrm{x}\, rot\,\vec{E}_a\right) dS = k_a^2 \int_V \left(\vec{H}_a^2 - \vec{E}_a^2\right) dV$$

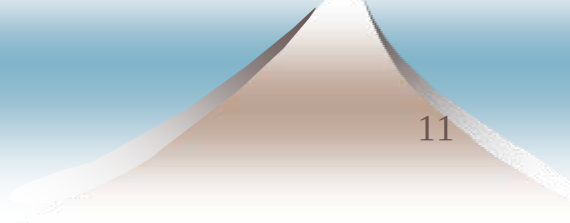
$$\int_V \vec{E}_a^2\, dV = \int_V \vec{H}_a^2\, dV$$

$$\int_V \vec{E}_a \cdot \vec{E}_b\, dV \equiv \delta_{ab}\ , \quad \int_V \vec{H}_a \cdot \vec{H}_b\, dV \equiv \delta_{ab}$$

Solenoidal Components

$$\vec{E} = \sum_a \mathrm{e}_a \vec{E}_a = \sum_a \vec{E}_a \int_V \vec{E}_a \cdot \vec{E}\, dV$$

$$\vec{H} = \sum_a \mathrm{h}_a \vec{H}_a = \sum_a \vec{H}_a \int_V \vec{H}_a \cdot \vec{H}\, dV$$



Normal Mode Analysis（Slator)

Normal Conductor

- Igen Value ; Frequency
- Igen Function ; Field Distribution $\vec{E}_a \, , \vec{H}_a$

$$\int_V \vec{E}_a \cdot \vec{E}_b \, dV = \delta_{ab} \; , \quad \int_V \vec{H}_a \cdot \vec{H}_b \, dV = \delta_{ab}$$

$$\vec{E} \boxed{\cong} \sum_a \mathrm{e}_a \vec{E}_a = \sum_a \vec{E}_a \int_V \vec{E}_a \cdot \vec{E} \, d\,V$$

$$\vec{H} \boxed{\cong} \sum_a \mathrm{h}_a \vec{H}_a = \sum_a \vec{H}_a \int_V \vec{H}_a \cdot \vec{H} \, d\,V$$

Time Dependence

$$rot\,\vec{E} = \sum_a \vec{H}_a \int_V rot\,\vec{E}\cdot\vec{H}_a\,dV\ , \quad rot\,\vec{H} = \sum_a \vec{E}_a \int_V rot\,\vec{H}\cdot\vec{E}_a\,dV\ ,$$

$$div\left(\vec{E}\,\mathrm{x}\,rot\,\vec{E}_a\right) = rot\,\vec{E}\cdot rot\,\vec{E}_a - \vec{E}\cdot rot\,rot\,\vec{E}_a = k_a\,\vec{H}_a\cdot rot\,\vec{E} - k_a^2\,\vec{E}\cdot\vec{E}_a$$

Integrate

$$\int_{S,S'} \vec{n}\cdot\left(\vec{E}\,\mathrm{x}\,\vec{H}_a\right)dS = \int_V rot\,\vec{E}\cdot\vec{H}_a\,dV - k_a\int_V \vec{E}\cdot\vec{E}_a\,dV$$

$$\int_V rot\,\vec{E}\cdot\vec{H}_a\,dV = k_a\int_V \vec{E}\cdot\vec{E}_a\,dV + \boxed{\int_S \vec{H}_a\cdot\left(\vec{n}\,\mathrm{x}\,\vec{E}\right)dS}$$

$$\int_V rot\,\vec{H}\cdot\vec{E}_a\,dV = k_a\int_V \vec{H}\cdot\vec{H}_a\,dV + \boxed{\int_{S'} \vec{E}_a\cdot\left(\vec{n}\,\mathrm{x}\,\vec{H}\right)dS}$$

Real Conductor

$$rot\,\vec{E} = \sum_a \vec{H}_a \left[k_a \int_V \vec{E}\cdot\vec{E}_a\, dV + \boxed{\int_S \left(\vec{n}\,\mathrm{x}\,\vec{E}\right)\cdot\vec{H}_a\, dS} \right]$$

$$rot\,\vec{H} = \sum_a \vec{E}_a \left[k_a \int_V \vec{H}\cdot\vec{H}_a\, dV + \boxed{\int_{S'} \left(\vec{n}\,\mathrm{x}\,\vec{H}\right)\cdot\vec{E}_a\, dS} \right]$$

$\Downarrow$ Maxwell’s Equation

$$\mu \frac{d}{dt} \int_V \vec{H}\bullet\vec{H}_a\, dV \cong -k_a \int_V \vec{E}\bullet\vec{E}_a\, dV - \boxed{\int_S \left(\vec{n}\,\mathrm{x}\,\vec{E}\right)\bullet\vec{H}_a\, dS}$$

$$\varepsilon \frac{d}{dt} \int_V \vec{E}\bullet\vec{E}_a\, dV \cong k_a \int_V \vec{H}\bullet\vec{H}_a\, dV - \int_V \vec{J}\bullet\vec{E}_a\, dV + \boxed{\int_{S'} \left(\vec{n}\,\mathrm{x}\,\vec{H}\right)\bullet\vec{E}_a\, dS}$$

Time Dependence

$$\varepsilon\mu\frac{d^2}{dt^2}\int_V \vec{E}\cdot\vec{E}_a\,dV + k_{\mathrm{a}}^2\int_V \vec{E}\cdot\vec{E}_a\,dV$$

$$= \underbrace{-\mu\frac{d}{dt}\left[\int_V \vec{J}\cdot\vec{E}_a\,dV\right.}_{\textbf{Dielectric Loss}} \underbrace{\left.-\int_{S'}\left(\vec{n}\,\mathrm{x}\,\vec{H}\right)\cdot\vec{E}_a\,dS\right]}_{\textbf{External Drive}} \underbrace{-k_a\int_S\left(\vec{n}\,\mathrm{x}\,\vec{E}\right)\cdot\vec{H}_a\,dS}_{\textbf{Wall Loss}}$$

$$\varepsilon\mu\frac{d^2}{dt^2}\int_V \vec{H}\cdot\vec{H}_a\,dV + k_{\mathrm{a}}^2\int_V \vec{H}\cdot\vec{H}_a\,dV$$

$$= k_a\left[\int_V \vec{J}\cdot\vec{E}_a\,dV - \int_{S'}\left(\vec{n}\,\mathrm{x}\,\vec{H}\right)\cdot\vec{E}_a\,dS\right] - \varepsilon\frac{d}{dt}\int_S\left(\vec{n}\,\mathrm{x}\,\vec{E}\right)\cdot\vec{H}_a\,dS$$

No Dielectric

$$\varepsilon\,\mu \frac{d^2}{d\,t^2}\,\mathrm{e}_a + k_a^2\,\mathrm{e}_a$$

$$= \mu \frac{d}{d\,t}\left[\int_{S'} \left(\vec{n}\,\mathrm{x}\,\vec{H}\right)\cdot\,\vec{E}_a\,dS\right] - k_a \int_S \left(\vec{n}\,\mathrm{x}\,\vec{E}\right)\cdot\,\vec{H}_a\,dS$$

$$\frac{d^2}{d\,t^2}\,\mathrm{e}_a + \omega_a^2\,\mathrm{e}_a$$

$$= \frac{1}{\varepsilon}\frac{d}{d\,t}\left[\int_{S'} \left(\vec{n}\,\mathrm{x}\,\vec{H}\right)\cdot\,\vec{E}_a\,dS\right] - \frac{\omega_a}{\sqrt{\varepsilon\,\mu}} \int_S \left(\vec{n}\,\mathrm{x}\,\vec{E}\right)\cdot\,\vec{H}_a\,dS$$

Wall Loss & Q

RF Surface Registance ; $R_S = \sqrt{\dfrac{\omega\mu}{2\sigma_{\text{wall}}}} = \dfrac{1}{\delta\sigma_{\text{wall}}}$

Wall Loss ; $P_0 = \dfrac{R_S}{2}\int_S \left|\vec{H}\right|^2 dS$

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma_{\text{wall}}}}$$

$$\vec{n}\,\mathrm{x}\,\vec{E} = (1+j)\sqrt{\frac{\omega\mu}{2\sigma_{\text{wall}}}}\,\vec{H} = (1+j)\,R_S\,\vec{H}$$

$$\int_V \vec{H}\cdot\vec{H}_a\,dV = j\sqrt{\frac{\varepsilon}{\mu}}\int_V \vec{E}\cdot\vec{E}_a\,dV$$

$$\vec{n} \, \mathrm{x} \, \vec{E} = Z_S \vec{H} = (R_S + j X_S) \vec{H}$$

$$\frac{d^2}{d\,t^2}\,\mathrm{e}_a + \omega_a^2\,\mathrm{e}_a$$

$$= \frac{1}{\varepsilon}\frac{d}{d\,t}\left[\int_{S'}\left(\vec{n}\,\mathrm{x}\,\vec{H}\right)\cdot\,\vec{E}_a\,dS\right] - \frac{\omega_a}{\sqrt{\varepsilon\,\mu}}\int_S\left(\vec{n}\,\mathrm{x}\,\vec{E}\right)\cdot\,\vec{H}_a\,dS$$

$$= \frac{1}{\varepsilon}\frac{d}{d\,t}\left[\int_{S'}\left(\vec{n}\,\mathrm{x}\,\vec{H}\right)\cdot\,\vec{E}_a\,dS\right] + \left(1-j\right)\omega_a\,\mathrm{e}_a\,\frac{R_S}{\mu}\int_S\left|\vec{H}_a\right|^2 dS$$

$$\Downarrow$$

$$Q_0 = \frac{\omega_a W}{P_0}\,,\quad W = \frac{\mu}{2} \Longrightarrow \left(1-j\right)\frac{\omega_a^2}{Q_0}\,\mathrm{e}_a$$

No External Drive

$$\frac{d^2}{dt^2}\mathrm{e}_a + \left(1-\frac{1-j}{Q_0}\right)\omega_a^2\,\mathrm{e}_a = 0$$

$$\mathrm{e}_a \propto \exp\left(j\omega t\right),\quad \omega = \omega_1 + j\omega_2$$

$$\omega_1 \cong \omega_a\left(1-\frac{1}{2Q_0}\right),\quad \omega_2 \cong \frac{\omega_a}{2Q_0}$$

$$\mathrm{e}_a \propto \exp\left(-\frac{\omega_a}{2Q_0}t\right)\exp\left(j\omega_1 t\right)$$

$$\varepsilon \mu \frac{d^2}{dt^2} \mathrm{h}_a + k_{\mathrm{a}}^2 \mathrm{h}_a$$

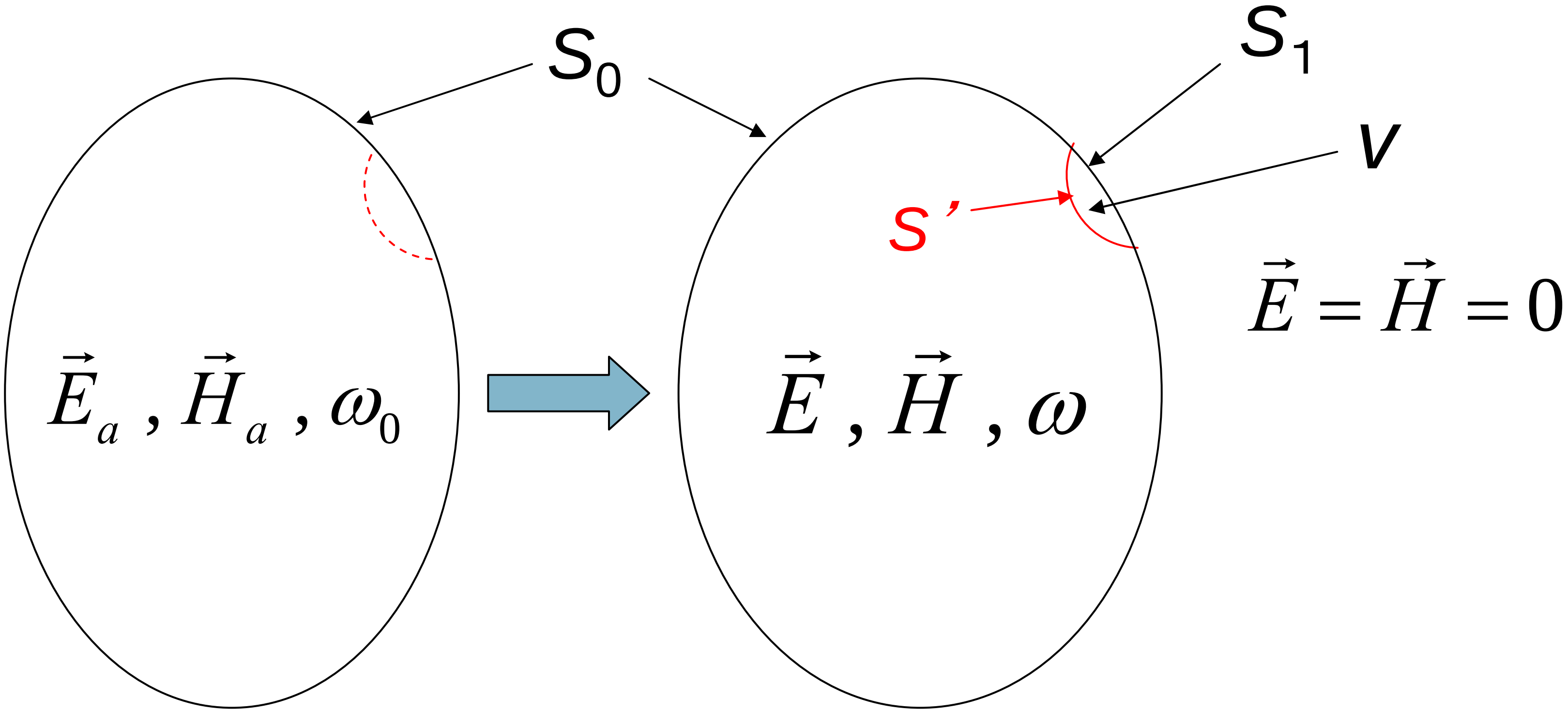
$$= -k_a \int_{S'} \left(\vec{n} \,\mathrm{x}\, \vec{H} \right) \cdot \vec{E}_a \, dS - \varepsilon \frac{d}{dt} \int_S \left(\vec{n} \,\mathrm{x}\, \vec{E} \right) \cdot \vec{H}_a \, dS$$

$$\frac{d^2}{dt^2} \mathrm{h}_a + \omega_{\mathrm{a}}^2 \mathrm{h}_a$$

$$= -\frac{\omega_a}{\sqrt{\varepsilon \mu}} \int_{S'} \left(\vec{n} \,\mathrm{x}\, \vec{H} \right) \cdot \vec{E}_a \, dS - \frac{1}{\mu} \frac{d}{dt} \int_S \left(\vec{n} \,\mathrm{x}\, \vec{E} \right) \cdot \vec{H}_a \, dS$$

$$= -\frac{\omega_a}{\sqrt{\varepsilon \mu}} \int_{S'} \left(\vec{n} \,\mathrm{x}\, \vec{H} \right) \cdot \vec{E}_a \, dS - \left(1 + j \right) \frac{\omega_a}{Q_0} \frac{d}{dt} \mathrm{h}_a$$

Perturbation



S_0

S_1

v

S'

$\vec{E} = \vec{H} = 0$

$\vec{E}_a, \vec{H}_a, \omega_0$

$\vec{E}, \vec{H}, \omega$

Perfect Conductor

$\vec{n} \times \vec{E}_a = 0, \quad \vec{n} \cdot \vec{H}_a = 0$

on S_0 and S_1

$\vec{n} \times \vec{E} = 0, \quad \vec{n} \cdot \vec{H} = 0$

on S_0 and S'

For Perturbed Cavity

$$\varepsilon\mu\frac{d^2}{dt^2}\int_V \vec{H}\cdot\vec{H}_a\,dV + k_{\mathrm{a}}^2\int_V \vec{H}\cdot\vec{H}_a\,dV$$

$$= k_a\left[\int_V \vec{J}\cdot\vec{E}_a\,dV - \int_{S'}\left(\vec{n}\,\mathrm{x}\,\vec{H}\right)\cdot\vec{E}_a\,dS\right] - \varepsilon\frac{d}{dt}\int_S\left(\vec{n}\,\mathrm{x}\,\vec{E}\right)\cdot\vec{H}_a\,dS$$

$$\Downarrow$$

$$\varepsilon\mu\frac{d^2}{dt^2}\int_V \vec{H}\cdot\vec{H}_a\,dV + k_{\mathrm{a}}^2\int_V \vec{H}\cdot\vec{H}_a\,dV$$

$$= -k_a\int_{S'}\left(\vec{n}\,\mathrm{x}\,\vec{H}\right)\cdot\vec{E}_a\,dS$$

$$\int_{S'} \left(\vec{n} \,\mathrm{x}\, \vec{H}\right) \cdot \vec{E}_a \, dS = -\int_{S'} \vec{n} \cdot \left(\vec{E}_a \,\mathrm{x}\, \vec{H}\right) dS \cong -\int_{S'} \vec{n} \cdot \left(\vec{E}_a \,\mathrm{x}\, \vec{H}_a\right) dS$$

$$= -\int_v div\left(\vec{E}_a \,\mathrm{x}\, \vec{H}_a\right) dV = -\int_v \left(\vec{H}_a \cdot rot\, \vec{E}_a - \vec{E}_a \cdot rot\, \vec{H}_a\right) dV$$

$$= -k_a \int_v \left(\vec{H}_a^2 - \vec{E}_a^2\right) dV$$

$$-\varepsilon\,\mu\,\omega^2 + k_a^2 = -\int_v \left(\vec{H}_a^2 - \vec{E}_a^2\right) dV$$

$$\omega^2 = \omega_0^2 \left(1 + \int_v \left(\vec{H}_a^2 - \vec{E}_a^2\right) dV \right)$$

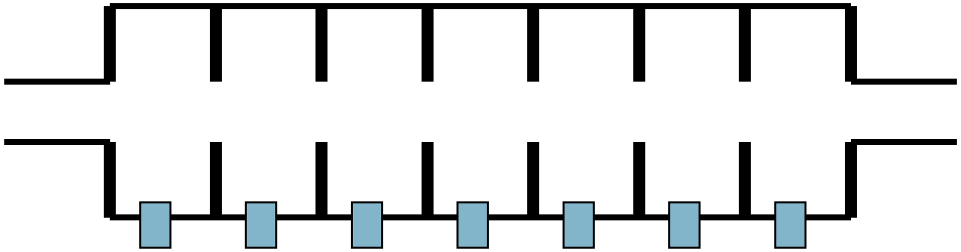
Perturbation

Volume Change of Δ V → Frequency Change of Δ f

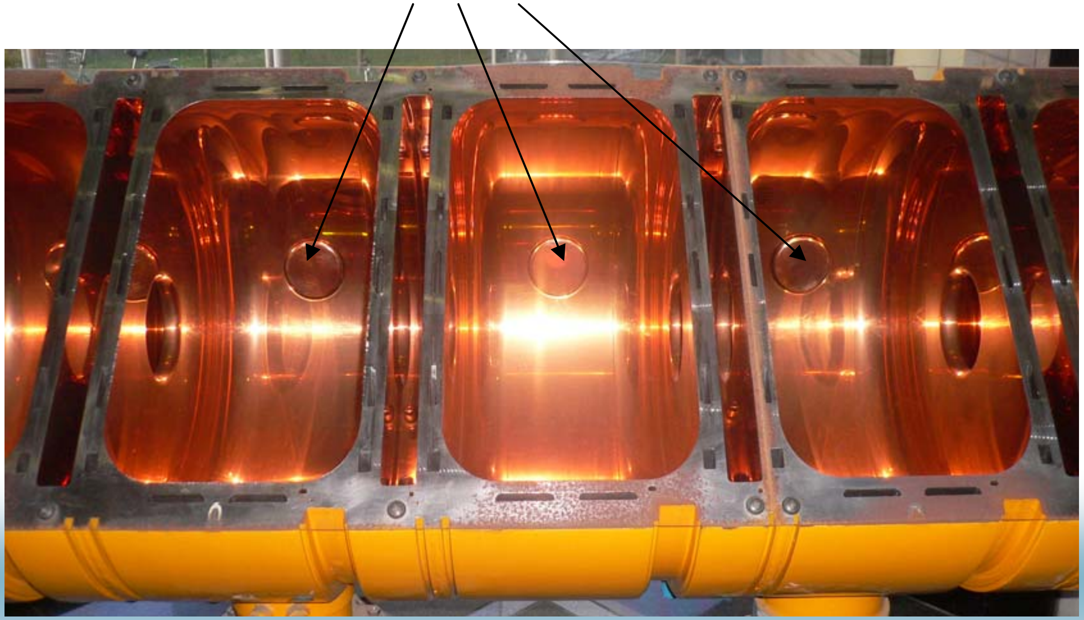
$$\frac{\omega^2}{\omega_0^2} = 1 + \frac{\int_{\Delta V} \left(\varepsilon \left| \vec{E} \right|^2 - \mu \left| \vec{H} \right|^2 \right) dV}{2W}$$

$$\frac{\delta f}{f_0} \approx \frac{1}{4W} \int_{\Delta V} \left(\varepsilon \left| \vec{E} \right|^2 - \mu \left| \vec{H} \right|^2 \right) dV$$

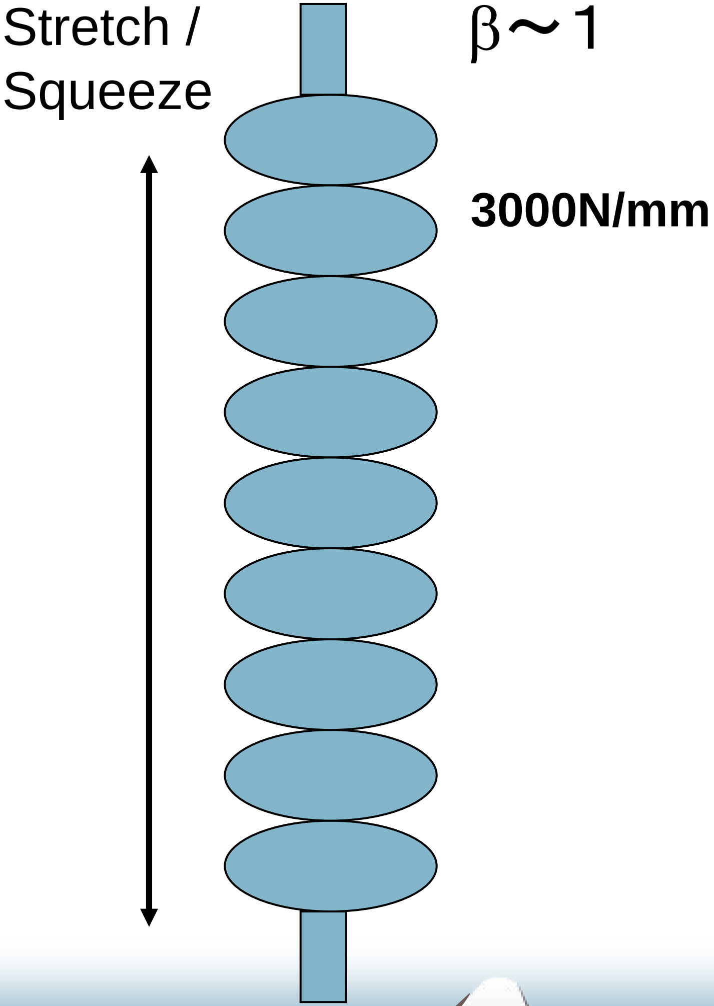
Normal Conducting



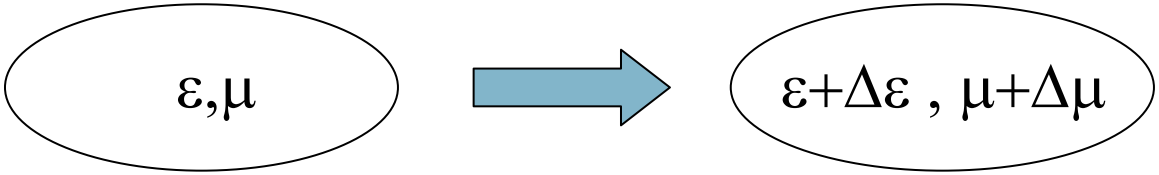
Piston Tuners



Superconducting



Change of ε , μ


ε,μ

ε+Δε , μ+Δμ

$$rot\,\vec{E}_0 = -j\,\omega_0\,\mu\,\vec{H}_0 \quad \Rightarrow \quad rot\,\vec{E} = -j\,\omega\left(\mu + \Delta\,\mu\right)\vec{H}$$

$$rot\,\vec{H}_0 = j\,\omega_0\,\varepsilon\,\vec{E}_0 \quad \Rightarrow \quad rot\,\vec{H} = j\,\omega\left(\varepsilon + \Delta\,\varepsilon\right)\vec{E}$$

$$j\,\omega\left(\varepsilon + \Delta\,\varepsilon\right)\vec{E}\cdot\vec{E}_0^* - j\,\omega_0\,\mu\,\vec{H}_0^*\cdot\vec{H} = \vec{E}_0^*\cdot rot\,\vec{H} - \vec{H}\cdot rot\,\vec{E}_0^*$$

$$= div\left(\vec{H}\ \mathrm{x}\ \vec{E}_0^*\right)$$

$$j\,\omega\left(\mu + \Delta\,\mu\right)\vec{H}\cdot\vec{H}_0^* - j\,\omega_0\,\varepsilon\,\vec{E}_0^*\cdot\vec{E} = -\vec{H}_0^*\cdot rot\,\vec{E} + \vec{E}\cdot rot\,\vec{H}_0^*$$

$$= div\left(\vec{H}_0^*\ \mathrm{x}\ \vec{E}\right)$$

$$\int_V \left[\{\omega(\varepsilon+\Delta\varepsilon)-\omega_0\,\varepsilon\}\vec{E}\cdot\vec{E}_0^* + \{\omega(\mu+\Delta\mu)-\omega_0\,\mu\}\vec{H}\cdot\vec{H}_0^*\right]dV$$

$$=\int_S \vec{n}\cdot\left(\vec{H}\,\mathrm{x}\,\vec{E}_0^* + \vec{H}_0^*\,\mathrm{x}\,\vec{E}\right)dS$$

$$=-\int_S \left[\vec{H}\cdot\left(\vec{n}\,\mathrm{x}\,\vec{E}_0^*\right)+\vec{H}_0^*\cdot\left(\vec{n}\,\mathrm{x}\,\vec{E}\right)\right]dS=0$$

$$\int_V \left[\left(\omega_0\Delta\varepsilon+\delta\omega\varepsilon\right)\vec{E}\cdot\vec{E}_0^* + \left(\omega_0\Delta\mu+\delta\omega\mu\right)\vec{H}\cdot\vec{H}_0^*\right]dV=0$$

$$\omega_0\int_V \left(\Delta\varepsilon\vec{E}\cdot\vec{E}_0^* + \Delta\mu\,\vec{H}\cdot\vec{H}_0^*\right)dV=-\delta\omega\int_V\left(\varepsilon\vec{E}\cdot\vec{E}_0^* + \mu\,\vec{H}\cdot\vec{H}_0^*\right)dV$$

$$\frac{\delta\omega}{\omega_0}=-\frac{\int_V\left(\Delta\varepsilon\vec{E}\cdot\vec{E}_0^* + \Delta\mu\,\vec{H}\cdot\vec{H}_0^*\right)dV}{2W}$$

Field Distribution Measurement

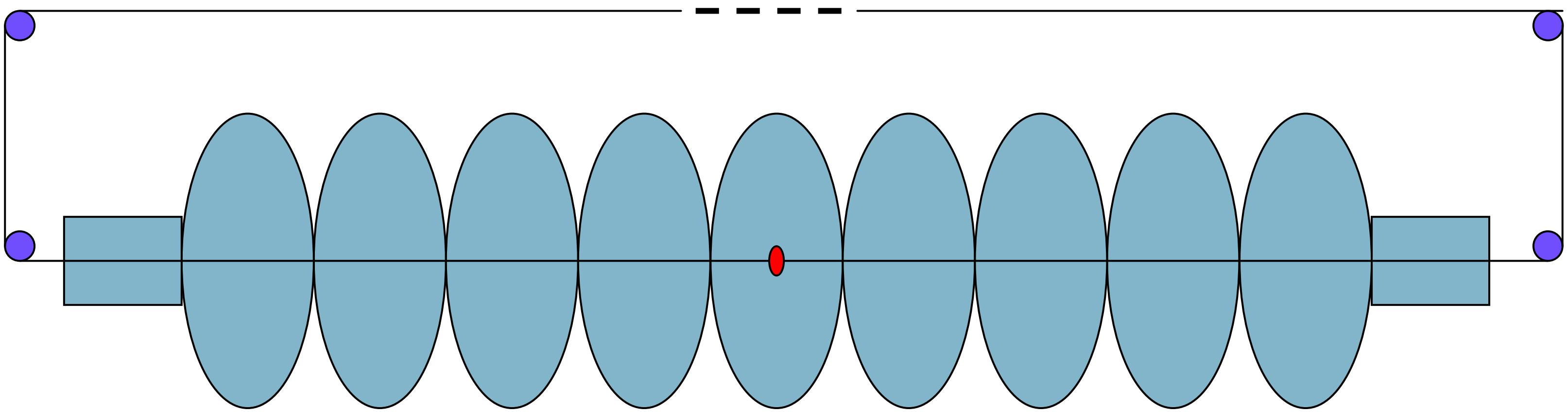
Metal ; Volume Change, Dielectric / Magnetic ; Change of ε,μ

$$\frac{\Delta f}{f_0} \approx -\frac{\mathrm{X}_{\mathrm{er}}\left|\vec{E}\right|^2 + \mathrm{X}_{\mathrm{mr}}\left|\vec{H}\right|^2}{2W}\Delta V$$

$$\mathrm{X}_{\mathrm{er}} = \frac{3\varepsilon_0(\varepsilon-1)}{\varepsilon+2}, \quad \mathrm{X}_{\mathrm{mr}} = \frac{3\mu_0(\mu-1)}{\mu+2}; \quad \text{Sphere}$$

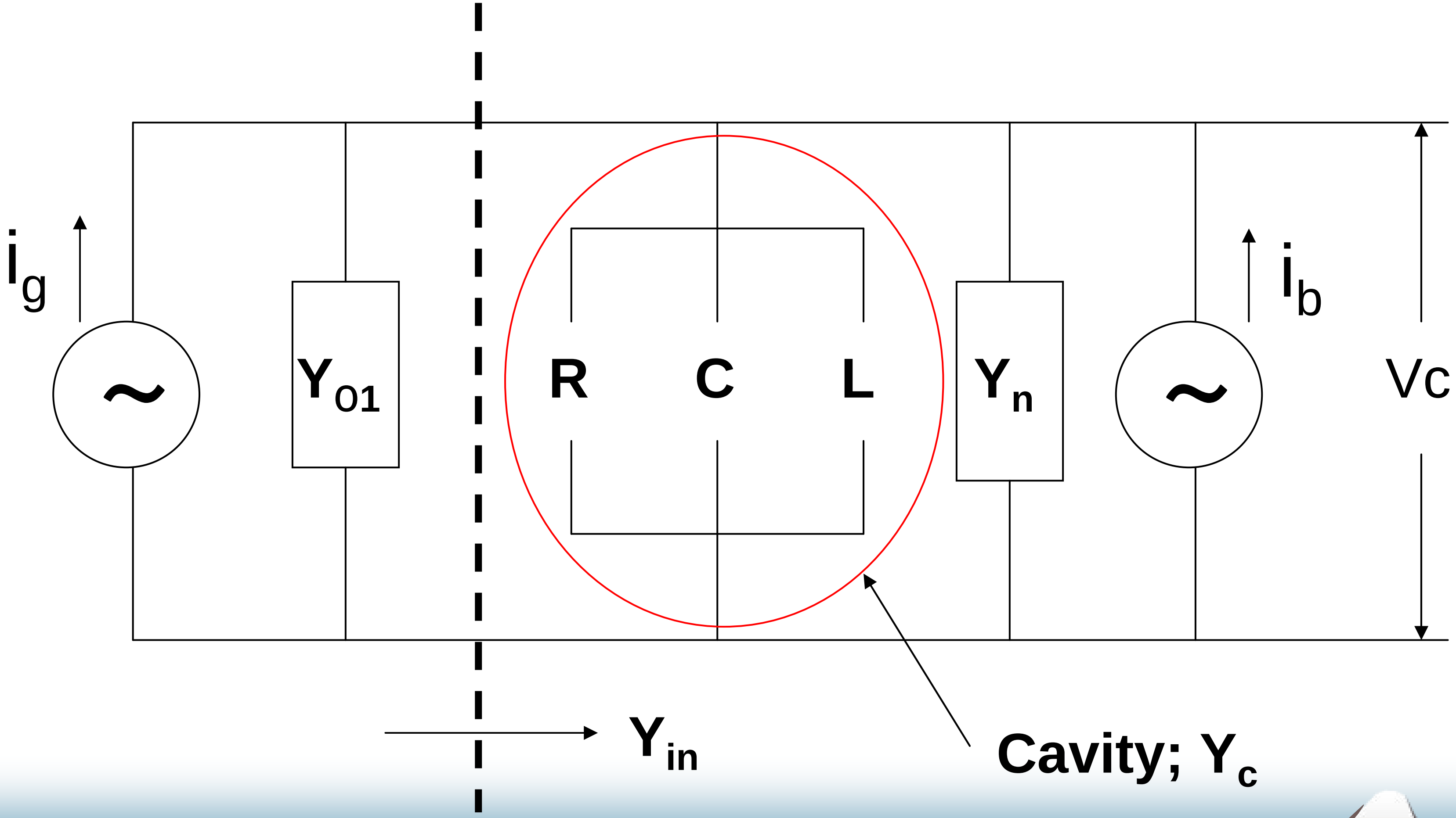
$$\mathrm{X}_{\mathrm{er}} = \varepsilon_0(\varepsilon-1), \quad \mathrm{X}_{\mathrm{mr}} = \mu_0(\mu-1)\ ; \quad \text{Needle}$$

Beads pull Method



Powering the Cavity

Equivalent Circuit



Cavity ; Parallel L,C,R Circuit

$$\frac{d^2}{dt^2}V + \frac{1}{RC}\frac{d}{dt}V + \frac{1}{LC}V = 0$$

$$\omega_0^2 = \frac{1}{LC}, \quad Q_0 = R\sqrt{\frac{C}{L}} = RC\omega_0$$

$$\frac{d^2}{dt^2}V + \frac{\omega_0}{Q_0}\frac{d}{dt}V + \omega_0^2 V = 0$$

Available Power

$$P_g = \frac{i_g^2}{8\,Y_{01}}, \quad i_g = 2\sqrt{2\,P_g\,Y_{01}}$$

Cavity Voltage

$$V_g = \frac{i_g}{Y_{01} + Y_C + \sum_n Y_n} = \frac{2\sqrt{2\,P_g\,Y_{01}}}{Y_{01} + Y_C + \sum_n Y_n}$$

$$Y_C = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right) = \frac{1}{R} + j\frac{Q_0}{R}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

$$\approx \frac{1}{R} + 2j\frac{Q_0}{R}\left(\frac{\omega - \omega_0}{\omega_0}\right)$$

$$Y_{01} + Y_C + \sum\nolimits_n Y_n = \frac{Q_0}{R}\left(\frac{R}{Q_0}Y_{01} + \sum\nolimits_n \frac{R}{Q_0}Y_n + \frac{1}{Q_0}\right) + 2j\frac{Q_0}{R}\left(\frac{\omega - \omega_0}{\omega_0}\right)$$

$$Q_{in} = \frac{Q_0}{RY_{01}}, \quad Q_n = \frac{Q_0}{RY_n}, \quad \frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_{in}} + \sum\nolimits_n \frac{1}{Q_n}$$

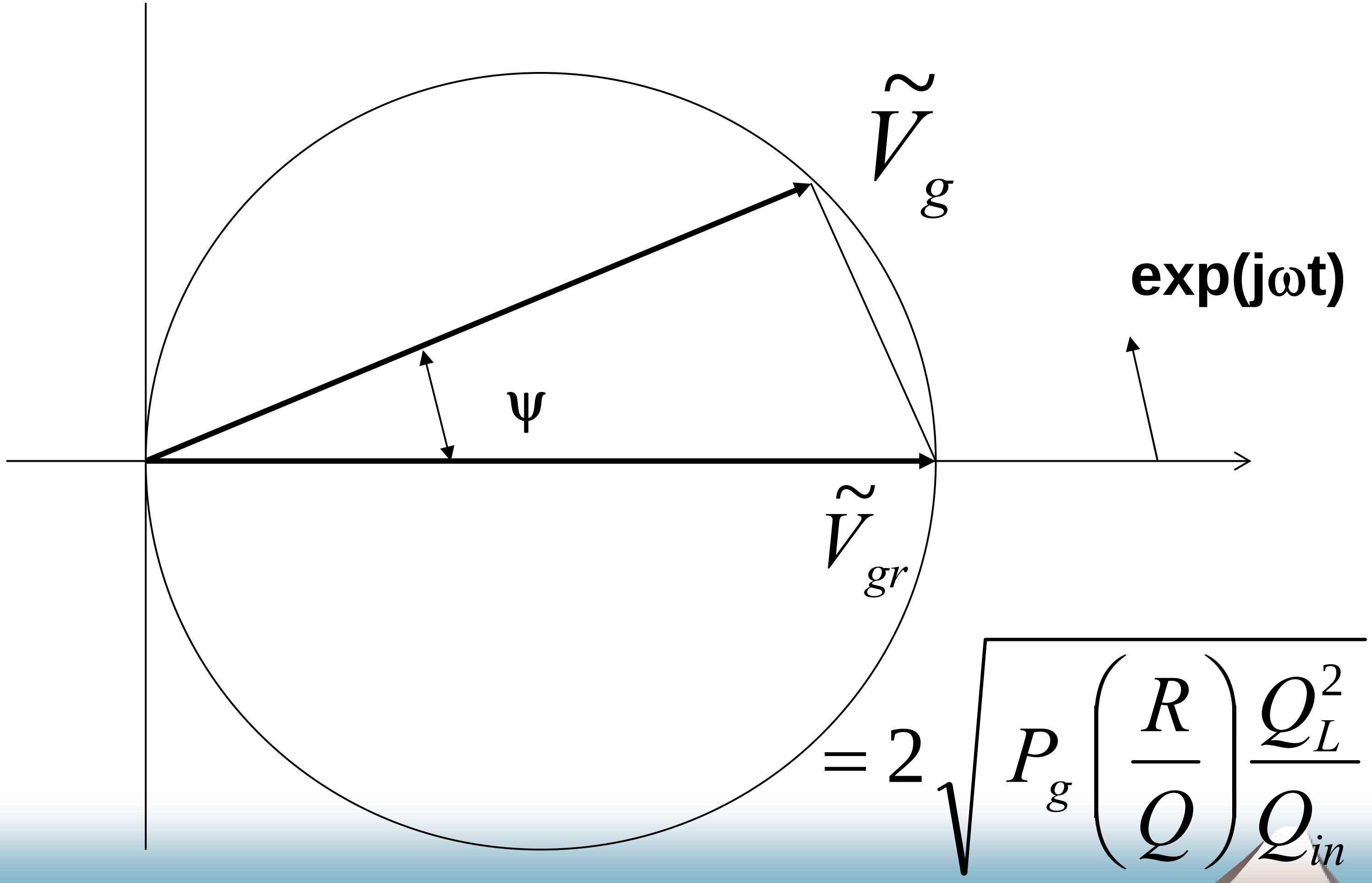
$$Y_{01} + Y_C + \sum_n Y_n = \frac{Q_0}{R\,Q_L} + 2\,j\,\frac{Q_0}{R}\left(\frac{\omega - \omega_0}{\omega_0}\right) = \frac{Q_0}{R\,Q_L}\left(1 - j\tan\psi\right)$$

$$\tan\psi \equiv -2\,Q_L\,\frac{\omega - \omega_0}{\omega_0}\,, \quad \psi\text{ ; Detuning Angle}$$

$$V_g = \frac{i_g}{Y_{01} + Y_C + \sum_n Y_n} = \frac{2\sqrt{2\,P_g\,Y_{01}}\,\dfrac{R\,Q_L}{Q_0}}{1 - j\tan\psi}$$

$$= 2\sqrt{2\,P_g\,\frac{R\,Q_L^2}{Q_0\,Q_{in}}}\cos\psi\,\exp\left(j\psi\right) = 2\sqrt{P_g\left(\frac{R}{Q}\right)\frac{Q_L^2}{Q_{in}}}\cos\psi\,\exp\left(j\psi\right)$$

Phaser Diagram



$$= 2\sqrt{P_g\left(\frac{R}{Q}\right)\frac{Q_L^2}{Q_{in}}}$$

Transmission

$$i_n = \frac{Y_n\, i_g}{Y_{01} + \sum_n Y_n + \frac{1}{R} + 2j\frac{Q_0}{R}\left(\frac{\omega - \omega_0}{\omega_0}\right)}, \quad P_n = \frac{\left|i_n\right|^2}{2Y_n}$$

$$T_n = \frac{P_n}{P_g} = \frac{4Y_{01}Y_n}{\left(Y_{01} + \sum_n Y_n + \frac{1}{R}\right)^2 + 4\left(\frac{Q_0}{R}\right)^2\left(\frac{\omega - \omega_0}{\omega_0}\right)^2}$$

$$= \frac{4\frac{R}{Q_0}Y_{01}\frac{R}{Q_0}Y_n}{\left(\frac{R}{Q_o}Y_{01} + \frac{R}{Q_0}\sum_n Y_n + \frac{1}{Q_0}\right)^2 + 4\left(\frac{\omega - \omega_0}{\omega_0}\right)^2}$$

$$T_n = \frac{P_n}{P_g} = \frac{4\dfrac{1}{Q_{in}}\dfrac{1}{Q_n}}{\left(\dfrac{1}{Q_{in}} + \dfrac{1}{Q_0} + \sum_n \dfrac{1}{Q_n}\right)^2 + 4\left(\dfrac{\omega - \omega_0}{\omega_0}\right)^2},$$

$$= \frac{4Q_L^2}{Q_{in}Q_n}\frac{1}{1 + 4Q_L^2\left(\dfrac{\omega - \omega_0}{\omega_0}\right)^2}$$

Band Width ; $T_n = 0.5$

Reflection

$$\Gamma = \frac{1 - Y_{in} / Y_{01}}{1 + Y_{in} / Y_{01}}$$

$$Y_{in} = \frac{1}{R} + 2j\frac{Q_0}{R}\left(\frac{\omega - \omega_0}{\omega_0}\right) + \sum_n Y_n$$

$$\beta_{in} = \frac{Q_0}{Q_{in}}, \quad \beta_n = \frac{Q_0}{Q_n}$$

$$\frac{Y_{in}}{Y_{01}} = \frac{1 + \sum_n \beta_n}{\beta_{in}} + 2j\left(\frac{\omega - \omega_0}{\omega_0}\right)Q_{in}$$

$$\beta^* \equiv \frac{1 + \sum_n \beta_n}{\beta_{in}}$$

$$|\Gamma| = \sqrt{\frac{P_{ref}}{P_g}} = \left|\frac{1-\beta^*}{1+\beta^*}\right|, \quad \text{at Resonance}$$

$$\beta^* = \frac{1+|\Gamma|}{1-|\Gamma|}, \quad \text{Over Coupling}$$

$$\beta^* = \frac{1-|\Gamma|}{1+|\Gamma|}, \quad \text{Under Coupling}$$

Cavity Measurement

$$\text{Decay}\,;\,W \propto \exp\left(-\frac{\omega_0}{Q_L}t\right), \quad \text{Band Width}\,;\,Q_L = \frac{\omega_0}{2\delta\omega} \qquad Q_L$$

$$P_g\,,P_r\,,P_n\,, \qquad P_o\,,\beta^*\,,\beta_n\,,\beta_{in}\,,Q_o\,,Q_{in}\,,Q_n$$

$$E_{acc} = \frac{1}{L_{Cavity}}\sqrt{\left(\frac{R}{Q}\right)\omega W} = \frac{1}{L_{Cavity}}\sqrt{\left(\frac{R}{Q}\right)P_n Q_n}$$

Cavity Voltage Equation

$$\frac{d^2}{dt^2}V+\left(1-\frac{1-j}{Q_0}\right)\omega_a^2 V=0\,,\quad \omega\cong\omega_a\left(1-\frac{1-j}{2Q_0}\right)$$

$$\frac{d^2}{dt^2}V+(1+j)\frac{\omega_a}{Q_0}\frac{d}{dt}V+\omega_a^2 V=0\,,\quad \omega\cong\omega_a\left(1-\frac{1-j}{2Q_0}\right)$$

$$\frac{d^2}{dt^2}V+\frac{\omega_a}{Q_0}\frac{d}{dt}V+\omega_a^2 V=0\,,\quad \omega\cong\omega_a\left(1+\frac{j}{2Q_0}\right)$$

Cavity Voltage Equation

$$\frac{d^2}{dt^2}\vec{V}(t)+(\frac{1}{Q_L}+j\frac{1}{Q_o})\omega_0\frac{d}{dt}\vec{V}(t)+\omega_o^2\vec{V}(t)=U(t)$$

Loss

Skin Depth

Drive Force

$\propto \exp(j\omega t)$

$$\vec{V} = \tilde{V}\exp(j\omega t)$$

$$\tilde{V} = \tilde{V}_d + (\tilde{V}_o - \tilde{V}_d)\exp(-\frac{t}{T_F})\exp(-j\frac{\tan\psi}{T_F}t)$$

$$T_F \equiv \frac{2Q_L}{\omega_0}; \quad \text{Filling Time}$$

$$\tan\psi \equiv -2Q_L\left(\frac{\omega-\omega_0}{\omega_0}\right); \quad \text{Detuning Angle}(\psi)$$

Cavity Voltage

at the CW Limit with Beam

$$\tilde{V} = \left[2\sqrt{P_g \left(\frac{R}{Q}\right) Q_o \frac{\beta}{(1+\beta)^2}} \exp(j\theta) - I_b \left(\frac{R}{Q}\right) Q_o \frac{1}{1+\beta} \right] \cos\psi \exp(j\psi)$$

Feedback works fine.

Soluton for Homogeneou s Equation

$$\vec{V} = \vec{A}\exp\left(\gamma t\right)$$

$$\gamma = -\left(1 + j\frac{Q_L}{Q_0}\right)\frac{\omega_0}{2Q_L} \pm j\omega_0\sqrt{1 - \left(\frac{1}{2Q_L} + j\frac{1}{2Q_0}\right)^2}$$

$$\approx -\frac{\omega_0}{2Q_L} \pm j\omega_0$$

Paticuler Solution

$$\vec{\mathrm{V}} = \vec{V_d}\exp\left(j\omega t\right)$$

$$\vec{V_d} = \frac{\mathrm{U}_0}{\omega_0^2 - \omega^2 - \frac{\omega\omega_0}{Q_0} + j\frac{\omega\omega_0}{Q_L}}$$

General Solution

$$\vec{V} = \vec{A}\exp\left(j\omega_0 t\right)\exp\left(-\frac{\omega_0}{2Q_L}t\right) + \vec{V_d}\exp\left(j\omega t\right)$$

$$\vec{V} = 0; \quad \text{at } t = 0 \quad then \quad \vec{A} = -\vec{V_d}$$

$$\vec{V} = \vec{V_d}\left\{1 - \exp\left(-\frac{\omega_0}{2Q_L}t\right)\exp\left(-j\frac{\tan\psi}{T_F}t\right)\right\}\exp\left(j\omega t\right)$$

Cavity Voltage

During Build-up

$$\vec{V} = 2\sqrt{P_g\left(\frac{R}{Q}\right)Q_o\,\frac{\beta}{(1+\beta)^2}}\left[1-\exp\left(-\frac{t}{T_F}\right)\exp\left(j\frac{\tan\psi}{T_F}t\right)\right]\cos\psi\,\exp\{j(\theta+\psi)\}$$

$$\vec{V} = \vec{V}_f \quad \text{at Beam Timing } T_e = T_F\,\text{In}\frac{1+\beta+\beta_b}{\beta_b}$$

$$V_{gr} = \frac{i_g}{Y_{01} + Y_C + \sum_n Y_n} = \frac{2\sqrt{2P_g Y_{01}}}{Y_{01} + Y_C + \sum_n Y_n}$$

$$V_g = \frac{V_{gr}}{1 + jQ_L\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)} = \frac{V_{gr}}{1 + j\tan\psi} = V_{gr}\cos\psi\exp(j\psi)$$