RF Basics; Contents

- Maxwell's Equation
- Plane Wave
- Boundary Condition
- Cavity & RF Parameters
- Normal Mode Analysis
- Perturbation Theory
- Equivalent Circuit
- Coupled Cavity

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Literatures

Maxwell's Equation

$$rot\,\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0\,, \quad div\,\vec{B} = 0,$$

$$rot\,\vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}\,, \quad div\,\vec{D} = \rho$$

$$\vec{D} = \varepsilon\,\vec{E}\,, \quad \vec{B} = \mu\,\vec{H}\,, \quad \boxed{\vec{J} = \sigma\,\vec{E}}$$

Not a Beam Current

$$rot\left(rot\vec{E}\right) = -\mu\frac{\partial}{\partial t}\left(rot\vec{H}\right), \quad \frac{\partial}{\partial t}\left(rot\vec{H}\right) = \frac{\partial}{\partial t}\vec{J} + \varepsilon\frac{\partial^2}{\partial t^2}\vec{E}$$

$$\varepsilon\mu\frac{\partial^2}{\partial t^2}\vec{E} + \mu\frac{\partial}{\partial t}\vec{J} = -rot\left(rot\vec{E}\right) = \nabla^2\vec{E} - \nabla\left(div\vec{E}\right) = \nabla^2\vec{E}$$

$$\nabla^2\vec{A} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\vec{A};$$

Cartesian Coordinate

$$= \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}\right]\vec{A};$$

Cylindrical Coordinate

Wave Equation $\quad \rho, \sigma = 0$

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0, \quad \nabla^2 \vec{H} - \frac{1}{c^2}\frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

$$\vec{E}, \vec{H} \propto e^{j\omega t}, \quad k^2 = \omega^2 \varepsilon \mu = \left(\frac{\omega}{c}\right)^2$$

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0, \quad \nabla^2 \vec{H} + k^2 \vec{H} = 0$$

Energy & Power Flow

From Maxwell's Equation

$$\int_V \vec{E}\cdot\vec{J}\,dV + \frac{\partial}{\partial t}\int_V \frac{1}{2}\left(\varepsilon\vec{E}^2 + \mu\vec{H}^2\right)dV = \int_V \left(\vec{E}\,rot\,\vec{H} - \vec{H}\,rot\,\vec{E}\right)dV$$

$$= -\int_V div\left(\vec{E}\,\mathrm{x}\,\vec{H}\right)dV = -\int_S \left(\vec{E}\,\mathrm{x}\,\vec{H}\right)_n dS$$

Energy Loss + Change of Electric and Magnetic Energy
= Power Flow at Boundary S

Pointing Vector ; $\vec{S} = \vec{E}\,\mathrm{x}\,\vec{H}$

Power Flow ; $P = \frac{1}{2}\mathrm{Re}\left(\vec{E}\,\mathrm{x}\,\vec{H}^*\right)$

Maxwell's Equation in Cartesian Coordinates

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x \,, \qquad \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x + j\omega\varepsilon E_x$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \,, \qquad \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y + j\omega\varepsilon E_y$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \,, \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z + j\omega\varepsilon E_z$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\varepsilon} \,, \qquad \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0$$

Maxwell's Equation in Cylindrical Coordinates

$$\frac{1}{r}\frac{\partial E_z}{\partial \theta}-\frac{\partial E_\theta}{\partial z}=-j\omega\mu H_r\ ,\qquad \frac{1}{r}\frac{\partial H_z}{\partial \theta}-\frac{\partial H_\theta}{\partial z}=J_r+j\omega\varepsilon E_r$$

$$\frac{\partial E_r}{\partial z}-\frac{\partial E_z}{\partial r}=-j\omega\mu H_\theta\ ,\qquad \frac{\partial H_r}{\partial z}-\frac{\partial H_z}{\partial r}=J_\theta+j\omega\varepsilon E_\theta$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rE_\theta\right)-\frac{1}{r}\frac{\partial E_r}{\partial \theta}=-j\omega\mu H_z\ ,\qquad \frac{1}{r}\frac{\partial}{\partial r}\left(rH_\theta\right)-\frac{1}{r}\frac{\partial H_r}{\partial \theta}=J_z+j\omega\varepsilon E_z$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rE_r\right)+\frac{1}{r}\frac{\partial E_\theta}{\partial \theta}+\frac{\partial E_z}{\partial z}=\frac{\rho}{\varepsilon}\ ,\qquad \frac{1}{r}\frac{\partial}{\partial r}\left(rH_r\right)+\frac{1}{r}\frac{\partial H_\theta}{\partial \theta}+\frac{\partial H_z}{\partial z}=0$$

Plane Wave in Uniform Medium

$$\vec{E} \equiv \vec{E}(z)\exp(j\omega t); \quad \vec{H} \equiv \vec{H}(z)\exp(j\omega t); \quad \rho = 0$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x, \qquad \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x + j\omega\varepsilon E_x$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y, \qquad \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y + j\omega\varepsilon E_y$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z, \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z + j\omega\varepsilon E_z$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0, \qquad \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0$$

Plane Wave in Uniform Medium

$$\vec{E} \equiv \vec{E}(z)\exp(j\omega t); \;\; \vec{H} \equiv \vec{H}(z)\exp(j\omega t); \;\; \rho = 0$$

$$\frac{\partial E_x}{\partial z} = -j\omega\mu H_y , \qquad \frac{\partial H_x}{\partial z} = (\sigma + j\omega\varepsilon)E_y$$

$$\frac{\partial E_y}{\partial z} = j\omega\mu H_x , \qquad \frac{\partial H_y}{\partial z} = -(\sigma + j\omega\varepsilon)E_x$$

$$E_z = 0 , \qquad H_z = 0$$

Plane Wave

$$\frac{d^2 E_x}{d z^2} = j\omega\mu(\sigma + j\omega\varepsilon)E_x \,, \quad E_x = E_1 \exp(-\gamma z) + E_2 \exp(\gamma z)$$

$\gamma = \alpha + j\beta \,,$ **Propagation Constant**

$$\alpha = \omega\left[\frac{\varepsilon\mu}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} - 1\right)\right]^{\frac{1}{2}},$$

Attenuation Constant

$$\beta = \omega\left[\frac{\varepsilon\mu}{2}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} + 1\right)\right]^{\frac{1}{2}},$$

Phase Constant

$$H_y = \frac{-1}{j\omega\mu}\frac{d\,E_x}{d\,z} = \frac{1}{Z_i}\left[E_1 \exp(-\gamma z) - E_2 \exp(\gamma z)\right],$$

$$Z_i = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon - j(\sigma/\omega\varepsilon)}}$$

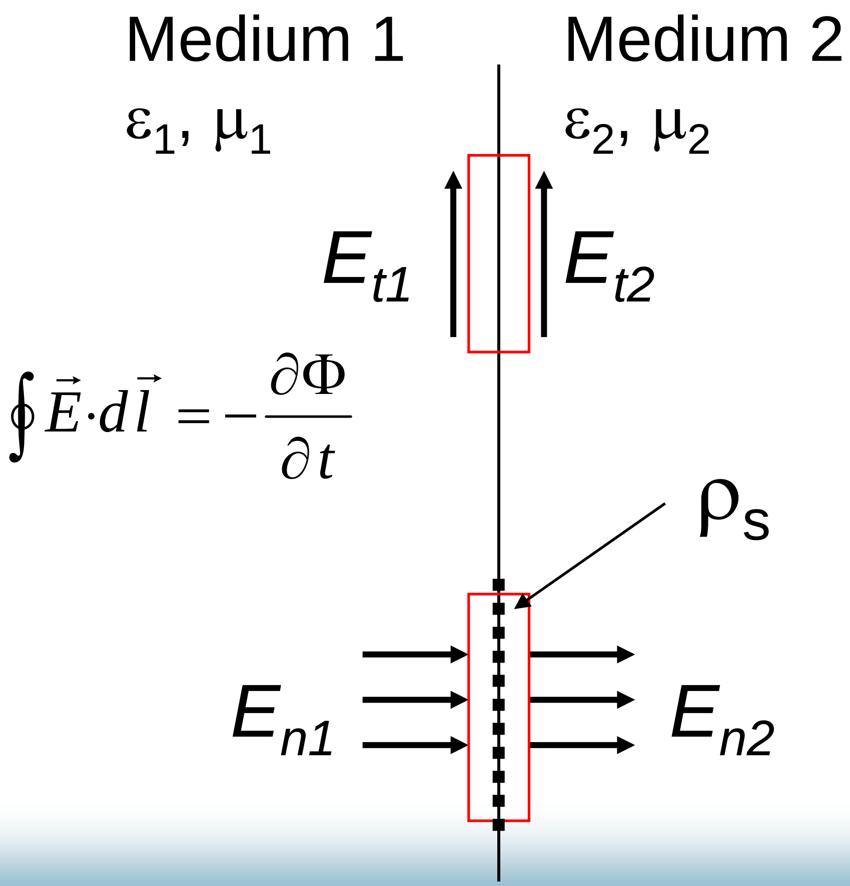
Intrinsic Impedance

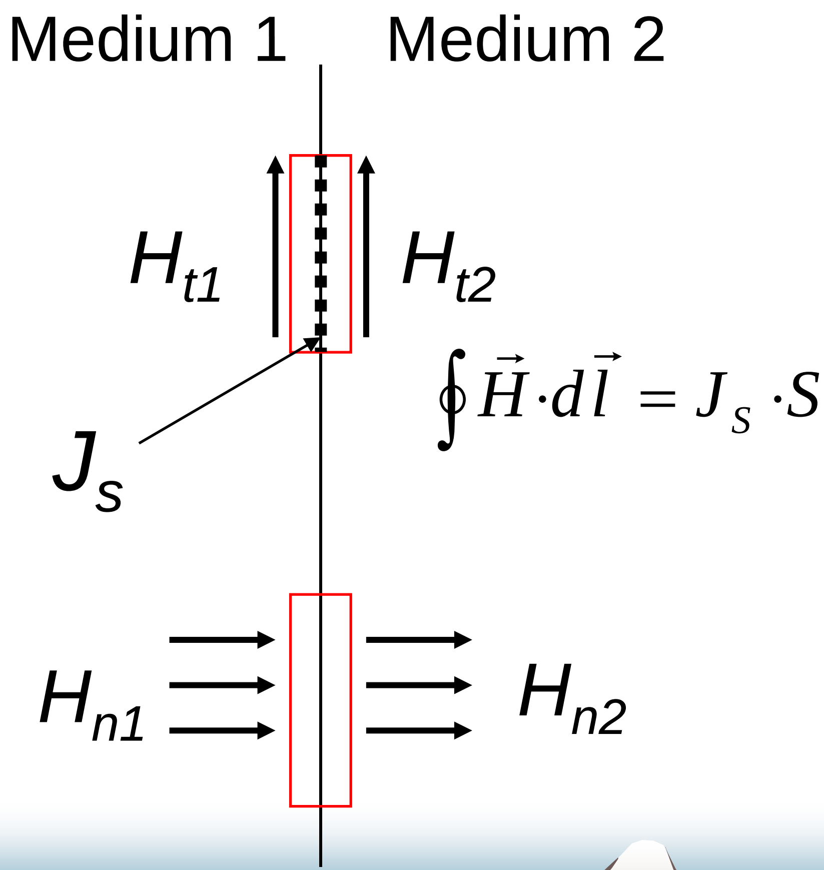
$\sigma >> \omega\varepsilon$; Conductor, $\gamma = (1+j)\sqrt{\frac{\omega\mu\sigma}{2}}$, $Z_i = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}}$

$\sigma << \omega\varepsilon$; Dielectric, $\gamma = j\omega\sqrt{\varepsilon\mu}$, $Z_i = \sqrt{\mu/\varepsilon}$

Boundary Condition

$$E_{t1} = E_{t2}\ ,\quad \varepsilon_2 E_{n2} - \varepsilon_1 E_{n1} = \rho_S\ ,\quad H_{t1} - H_{t2} = J_S\ ,\quad \mu_1 H_{n1} = \mu_2 H_{n2}$$





Reflection & Transmission

Medium 1 — ε_1, μ_1

Medium 2 — ε_2, μ_2

x

$$E_i e^{-jk_1 z}\ ,\ H_i = \frac{E_i}{Z_1} e^{-jk_1 z}$$

$$E_t e^{-jk_2 z}\ ,\ H_t = \frac{E_t}{Z_2} e^{-jk_2 z}$$

$$E_r e^{jk_1 z}\ ,\ H_r = -\frac{E_r}{Z_1} e^{jk_1 z}$$

z

$$E_{x1} = E_i e^{-jk_1 z} + E_r e^{jk_1 z} \;, \qquad E_{x2} = E_t e^{-jk_2 z}$$

$$H_{y1} = \frac{E_i}{Z_1} e^{-jk_1 z} - \frac{E_r}{Z_1} e^{jk_1 z} \;, \qquad H_{y2} = \frac{E_t}{Z_2} e^{-jk_2 z}$$

$$E_i + E_r = E_t \;, \qquad \frac{E_i - E_r}{Z_1} = \frac{E_t}{Z_2}$$

$$E_r = \frac{Z_2 - Z_1}{Z_2 + Z_1} E_i = \Gamma\, E_i \;, \qquad E_t = \frac{2 Z_2}{Z_2 + Z_1} E_i = T\, E_i$$

Metallic Boundary

$\sigma << \omega\varepsilon_1$

$Z_1 = \sqrt{\mu_1/\varepsilon_1}\ ,\ \beta_1 = \omega\sqrt{\varepsilon_1\mu_1}$

$\sigma_2 >> \omega\varepsilon_2$

$Z_2 = (1+j)\sqrt{\omega\mu_2/2\sigma_2}\ ,\ \alpha_2 + j\beta_2 = (1+j)\sqrt{\omega\mu_2\sigma_2/2}$

$$Z_1 >> |Z_2|, \quad E_r \cong -E_i\ , \quad E_t \cong 2\frac{Z_2}{Z_1}E_i$$

$$E_{x1} \cong -j2E_i\sin(k_1 z), \quad H_{y1} \cong \frac{2E_i}{Z_1}\cos(k_1 z)$$

$$E_{x2} \cong 2\frac{Z_2}{Z_1}E_i e^{-\alpha_2 z - j\beta_2 z}\ , \quad H_{y2} \cong 2\frac{E_i}{Z_1}e^{-\alpha_2 z - j\beta_2 z}$$

Metallic Boundary

Dielectric

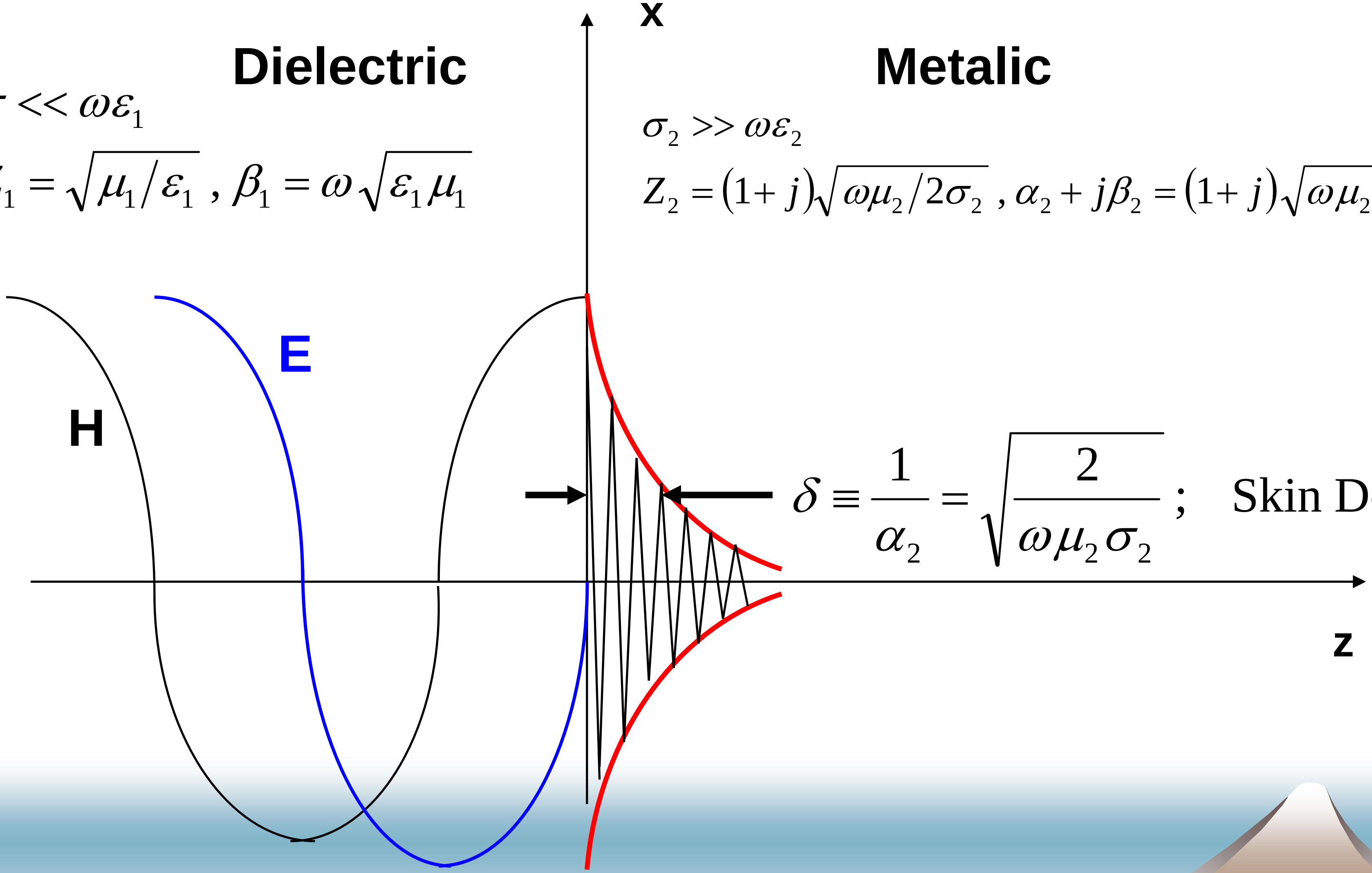
$\sigma << \omega\varepsilon_1$

$Z_1 = \sqrt{\mu_1/\varepsilon_1}\ ,\ \beta_1 = \omega\sqrt{\varepsilon_1\mu_1}$

Metalic

$\sigma_2 >> \omega\varepsilon_2$

$Z_2 = (1+j)\sqrt{\omega\mu_2/2\sigma_2}\ ,\ \alpha_2 + j\beta_2 = (1+j)\sqrt{\omega\mu_2\sigma_2/2}$



$$\delta \equiv \frac{1}{\alpha_2} = \sqrt{\frac{2}{\omega\mu_2\sigma_2}}\ ;\quad \text{Skin Depth}$$

Power Loss & Surface Impedance

$$P = \frac{1}{2}\mathrm{Re}\left(E_t \cdot H_t^*\right) = \frac{1}{2} H_t^2 \,\mathrm{Re}\left(Z_2\right) = \frac{1}{2} H_t^2 \sqrt{\frac{\omega\mu_2}{2\sigma_2}}$$

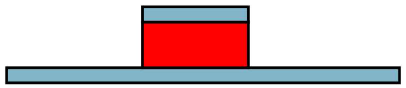
$Z_S = R_S + jX_S$; Surface Impedance

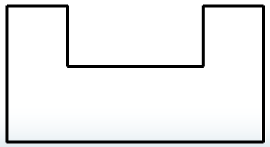
$R_S = \sqrt{\frac{\omega\mu_2}{2\sigma_2}} = \frac{1}{\delta\sigma_2}$; Surface Resistance

$P = \frac{1}{2} H_t^2 R_S$; Power Loss

Wave Guide

- Coaxial Line
- Parallel Conductor
- Strip Line
- Circular Wave Guide
- Rectangular Wave Guide
- Ridged Wave Guide





Traveling Wave Mode

$$\vec{E} \equiv \vec{E}(x,y)\exp(j\omega t - j\beta z);\; \vec{H} \equiv \vec{H}(x,y)\exp(j\omega t - j\beta z);\; \rho, \sigma = 0$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x\,, \qquad \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \cancel{J_x} + j\omega\varepsilon E_x$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y\,, \qquad \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \cancel{J_y} + j\omega\varepsilon E_y$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z\,, \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \cancel{J_z} + j\omega\varepsilon E_z$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \cancel{\frac{\rho}{\varepsilon}}\,, \qquad \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0$$

Traveling Wave Mode

$$\vec{E} \equiv \vec{E}(x, y)\exp(j\,\omega t - j\beta z);\;\; \vec{H} \equiv \vec{H}(x, y)\exp(j\,\omega t - j\beta z);\;\; \rho, \sigma = 0$$

$$j\,\beta\,E_y + \frac{\partial\,E_z}{\partial\,y} = -\,j\,\omega\,\mu\,H_x\,, \qquad j\,\beta\,H_y + \frac{\partial\,H_z}{\partial\,y} = \;j\,\omega\,\varepsilon\,E_x$$

$$-\,j\,\beta\,E_x - \frac{\partial\,E_z}{\partial\,x} = -\,j\,\omega\,\mu\,H_y\,, \qquad -\,j\,\beta\,H_x - \frac{\partial\,H_z}{\partial\,x} = \;j\,\omega\,\varepsilon\,E_y$$

$$\frac{\partial\,E_y}{\partial\,x} - \frac{\partial\,E_x}{\partial\,y} = -\,j\,\omega\,\mu\,H_z\,, \qquad \frac{\partial\,H_y}{\partial\,x} - \frac{\partial\,H_x}{\partial\,y} = \;j\,\omega\,\varepsilon\,E_z$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} - j\beta E_z = 0\,, \qquad \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} - j\beta H_z = 0$$

TE-Modes ; $E_z = 0$
Particular Solution

$$j\beta E_y = -j\omega\mu H_x \,, \qquad j\beta H_y + \frac{\partial H_z}{\partial y} = j\omega\varepsilon E_x$$

$$-j\beta E_x = -j\omega\mu H_y \,, \qquad -j\beta H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \,, \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0 \,, \qquad \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} - j\beta H_z = 0$$

$$H_x = -\frac{j\beta}{k^2-\beta^2}\frac{\partial H_z}{\partial x}, \qquad H_y = -\frac{j\beta}{k^2-\beta^2}\frac{\partial H_z}{\partial y}$$

$$E_x = -\frac{j\omega\mu}{k^2-\beta^2}\frac{\partial H_z}{\partial y}, \qquad E_y = -\frac{j\omega\mu}{k^2-\beta^2}\frac{\partial H_z}{\partial x}$$

$$\frac{\partial^2 H_z}{\partial x^2}+\frac{\partial^2 H_z}{\partial y^2}+k_c^2 H_z = 0, \quad k_c^2 \equiv k^2-\beta^2 = \omega^2\varepsilon\mu-\beta^2$$

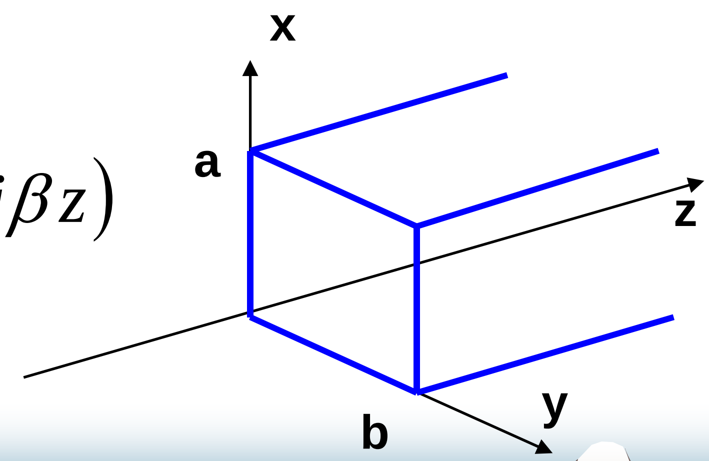
TE-mn Modes

$$H_z \equiv H_{z1}(x)H_{z2}(y), \quad \frac{1}{H_{z1}}\frac{d^2 H_{z1}}{d x^2} + \frac{1}{H_{z2}}\frac{d^2 H_{z2}}{d y^2} + k_c^2 = 0$$

$$\frac{d^2 H_{z1}}{d x^2} = -\beta_1^2 H_{z1}, \qquad \frac{d^2 H_{z2}}{d y^2} = -\beta_2^2 H_{z2}$$

$$k_c^2 = k^2 - \beta^2 = \beta_1^2 + \beta_2^2$$

$$H_z = \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\exp(-j\beta z)$$



$$\beta_1 = \frac{m\pi}{a}, \beta_2 = \frac{n\pi}{b}, m + n \geq 1$$

$$k_c = \sqrt{(m\pi/a)^2 + (n\pi/b)^2}\ ; \quad \beta = \sqrt{k^2 - k_c^2}$$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\omega\sqrt{\varepsilon\mu}}\ ;$$ **Wave Length in Medium**

$$\lambda_c = \frac{2\pi}{k_c} = 2 \Big/ \sqrt{(m/a)^2 + (n/b)^2}\ ;$$ **Critical Wave Length**

$$\lambda_g = \frac{2\pi}{\beta} = \lambda \Big/ \sqrt{1 - (\lambda/\lambda_c)^2}\ ;$$ **Guide Wave Length**

TE-mn Modes

$$E_x = \frac{j\omega\mu}{k_c^2}\frac{n\pi}{b}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}\ ,$$

$$E_y = -\frac{j\omega\mu}{k_c^2}\frac{m\pi}{a}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}\ ,$$

$$E_z = 0\ ,$$

$$H_x = \frac{j\beta}{k_c^2}\frac{m\pi}{a}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}\ ,$$

$$H_y = \frac{j\beta}{k_c^2}\frac{n\pi}{b}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}\ ,$$

$$H_z = \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

Loss

TM-Modes ; $H_z = 0$

$$j\beta E_y + \frac{\partial E_z}{\partial y} = -j\omega\mu H_x \,, \qquad j\beta H_y = j\omega\varepsilon E_x$$

$$-j\beta E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \,, \qquad -j\beta H_x = j\omega\varepsilon E_y$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \,, \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} - j\beta E_z = 0 \,, \qquad \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0$$

TM-mn Modes

$$E_x = -\frac{j\beta}{k_c^2}\frac{m\pi}{a}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}\ ,$$

$$E_y = -\frac{j\beta}{k_c^2}\frac{n\pi}{b}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}\ ,$$

$$E_z = \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}\ ,$$

$$H_x = \frac{j\omega\varepsilon}{k_c^2}\frac{n\pi}{b}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}\ ,$$

$$H_y = -\frac{j\omega\varepsilon}{k_c^2}\frac{m\pi}{a}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}\ ,$$

$$H_z = 0$$

TEM-Modes

$$j\beta E_y = -j\omega\mu H_x\,, \qquad j\beta H_y = j\omega\varepsilon E_x$$

$$-j\beta E_x = -j\omega\mu H_y\,, \qquad -j\beta H_x = j\omega\varepsilon E_y$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0\,, \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0\,, \qquad \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0$$

Maxwell's Equation in Cylindrical Coordinates

$$\frac{1}{r}\frac{\partial E_z}{\partial \theta}-\frac{\partial E_\theta}{\partial z}=-j\omega\mu H_r\ , \qquad \frac{1}{r}\frac{\partial H_z}{\partial \theta}-\frac{\partial H_\theta}{\partial z}=J_r+j\omega\varepsilon E_r$$

$$\frac{\partial E_r}{\partial z}-\frac{\partial E_z}{\partial r}=-j\omega\mu H_\theta\ , \qquad \frac{\partial H_r}{\partial z}-\frac{\partial H_z}{\partial r}=J_\theta+j\omega\varepsilon E_\theta$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rE_\theta\right)-\frac{1}{r}\frac{\partial E_r}{\partial \theta}=-j\omega\mu H_z\ , \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(rH_\theta\right)-\frac{1}{r}\frac{\partial H_r}{\partial \theta}=J_z+j\omega\varepsilon E_z$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rE_r\right)+\frac{1}{r}\frac{\partial E_\theta}{\partial \theta}+\frac{\partial E_z}{\partial z}=\frac{\rho}{\varepsilon}, \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(rH_r\right)+\frac{1}{r}\frac{\partial H_\theta}{\partial \theta}+\frac{\partial H_z}{\partial z}=0$$

Traveling Wave Modes

$$\vec{E} \equiv \vec{E}(r,\theta)\exp(j\omega t - j\beta z);\ \vec{H} \equiv \vec{H}(r,\theta)\exp(j\omega t - j\beta z);\ \rho, \sigma = 0$$

$$\frac{1}{r}\frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z} = -j\omega\mu H_r\ , \qquad \frac{1}{r}\frac{\partial H_z}{\partial \theta} - \frac{\partial H_\theta}{\partial z} = \cancel{J_r} + j\omega\varepsilon E_r$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -j\omega\mu H_\theta\ , \qquad \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = \cancel{J_\theta} + j\omega\varepsilon E_\theta$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) - \frac{1}{r}\frac{\partial E_r}{\partial \theta} = -j\omega\mu H_z\ , \qquad \frac{1}{r}\frac{\partial}{\partial r}(rH_\theta) - \frac{1}{r}\frac{\partial H_r}{\partial \theta} = \cancel{J_z} + j\omega\varepsilon E_z$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\frac{\partial E_\theta}{\partial \theta} + \frac{\partial E_z}{\partial z} = \frac{\cancel{\rho}}{\varepsilon}; \qquad \frac{1}{r}\frac{\partial}{\partial r}(rH_r) + \frac{1}{r}\frac{\partial H_\theta}{\partial \theta} + \frac{\partial H_z}{\partial z} = 0$$

Traveling Wave Modes

$$\vec{E} \equiv \vec{E}(r,\theta)\exp(j\omega t - j\beta z);\ \vec{H} \equiv \vec{H}(r,\theta)\exp(j\omega t - j\beta z);\ \rho, \sigma = 0$$

$$\frac{1}{r}\frac{\partial E_z}{\partial \theta} + j\beta E_\theta = -j\omega\mu H_r, \qquad \frac{1}{r}\frac{\partial H_z}{\partial \theta} + j\beta H_\theta = j\omega\varepsilon E_r$$

$$-j\beta E_r - \frac{\partial E_z}{\partial r} = -j\omega\mu H_\theta, \qquad -j\beta H_r - \frac{\partial H_z}{\partial r} = j\omega\varepsilon E_\theta$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) - \frac{1}{r}\frac{\partial E_r}{\partial \theta} = -j\omega\mu H_z, \qquad \frac{1}{r}\frac{\partial}{\partial r}(rH_\theta) - \frac{1}{r}\frac{\partial H_r}{\partial \theta} = j\omega\varepsilon E_z$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\frac{\partial E_\theta}{\partial \theta} - j\beta E_z = 0, \qquad \frac{1}{r}\frac{\partial}{\partial r}(rH_r) + \frac{1}{r}\frac{\partial H_\theta}{\partial \theta} - j\beta H_z = 0$$

TM-Modes ; $H_z = 0$

$$\frac{1}{r}\frac{\partial E_z}{\partial \theta} + j\beta E_\theta = -j\omega\mu H_r \,, \qquad j\beta H_\theta = j\omega\varepsilon E_r$$

$$-j\beta E_r - \frac{\partial E_z}{\partial r} = -j\omega\mu H_\theta \,, \qquad -j\beta H_r = j\omega\varepsilon E_\theta$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rE_\theta\right) - \frac{1}{r}\frac{\partial E_r}{\partial \theta} = 0 \,, \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(rH_\theta\right) - \frac{1}{r}\frac{\partial H_r}{\partial \theta} = j\omega\varepsilon E_z$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rE_r\right) + \frac{1}{r}\frac{\partial E_\vartheta}{\partial \theta} - j\beta E_z = 0 \,, \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(rH_r\right) + \frac{1}{r}\frac{\partial H_\vartheta}{\partial \theta} = 0$$

$$H_\theta = \frac{\omega\varepsilon}{\beta} E_r \,, \qquad H_r = -\frac{\omega\varepsilon}{\beta} E_\theta$$

$$E_r = -\frac{j\beta}{k^2-\beta^2}\frac{\partial E_z}{\partial r}\,, \qquad E_\theta = -\frac{j\beta}{k^2-\beta^2}\frac{1}{r}\frac{\partial E_z}{\partial \theta}$$

$$\frac{1}{r^2}\frac{\partial^2 E_z}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + k_c^2 E_z = 0\,, \qquad k_c^2 = k^2 - \beta^2$$

$$E_z = E_{z1}(r)\,E_{z2}(\theta);$$

$$\frac{d^2 E_{z2}}{d\,\theta^2} = -m^2\, E_{z2}\ ,$$

$$\frac{d^2 E_{z1}}{d\left(k_c\, r\right)^2} + \frac{1}{k_c\, r}\frac{d\,E_{z1}}{d\left(k_c\, r\right)} + \left(1 - \frac{m^2}{k_c^2\, r^2}\right) E_{z1} = 0\,,$$

$$E_{z2} = \cos m\theta\ ,$$

$$E_{z1} = J_m\left(k_c\, r\right); \quad \text{Bessel Function}$$

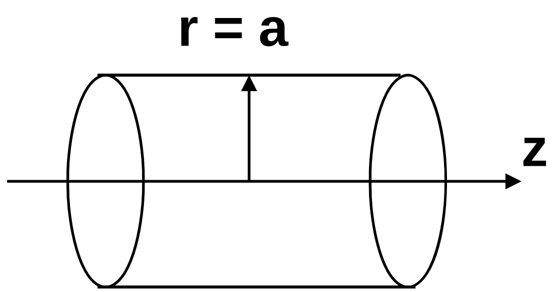
Boundary Condition

$$E_z(r=a)=0\,,$$

$$J_m(k_c\,a)=0\,,$$

$$k_c\,a = y_{mn}\,,$$

$$\beta = \sqrt{\omega^2\,\varepsilon\,\mu - \left(\frac{y_{mn}}{a}\right)^2}$$



y_{mn}

m n	1	2	3	4
0	2.4048	5.5201	8.6537	11.7915
1	3.8317	7.0156	10.1735	13.3237
2	5.1356	8.4172	11.6198	14.7960
3	6.3802	9.7610	13.0152	16.2235
4	7.5883	11.0647	14.3725	17.6160

TM-mn Modes

$$E_r = -j\frac{\beta}{(y_{mn}/a)}\cos m\theta J'_m\left(\frac{y_{mn}}{a}r\right)e^{-j\beta z} \ ,$$

$$E_\theta = j\frac{\beta m}{(y_{mn}/a)^2}\sin m\theta \frac{1}{r}J_m\left(\frac{y_{mn}}{a}r\right)e^{-j\beta z},$$

$$E_z = \cos m\theta \ J_m\left(\frac{y_{mn}}{a}r\right)e^{-j\beta z} \ ,$$

$$H_r = -j\frac{\omega\varepsilon m}{(y_{mn}/a)^2}\sin m\theta \frac{1}{r}J_m\left(\frac{y_{mn}}{a}r\right)e^{-j\beta z} \ ,$$

$$H_\theta = -j\frac{\omega\varepsilon}{(y_{mn}/a)}\cos m\theta J'_m\left(\frac{y_{mn}}{a}r\right)e^{-j\beta z} \ ,$$

$$H_z = 0$$

TE-Modes

$$j\beta E_\theta = -j\omega\mu H_r \,, \qquad \frac{1}{r}\frac{\partial H_z}{\partial \theta} + j\beta H_\theta = j\omega\varepsilon E_r$$

$$-j\beta E_r = -j\omega\mu H_\theta \,, \qquad -j\beta H_r - \frac{\partial H_z}{\partial r} = j\omega\varepsilon E_\theta$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rE_\theta\right) - \frac{1}{r}\frac{\partial E_r}{\partial \theta} = -j\omega\mu H_z \,, \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(rH_\theta\right) - \frac{1}{r}\frac{\partial H_r}{\partial \theta} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rE_r\right) + \frac{1}{r}\frac{\partial E_\theta}{\partial \theta} = 0 \,, \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(rH_r\right) + \frac{1}{r}\frac{\partial H_\theta}{\partial \theta} - j\beta H_z = 0$$

TEM-Modes ; $E_z = H_z = 0$

$$j\beta E_\theta = -j\omega\mu H_r \,, \qquad j\beta H_\theta = j\omega\varepsilon E_r$$

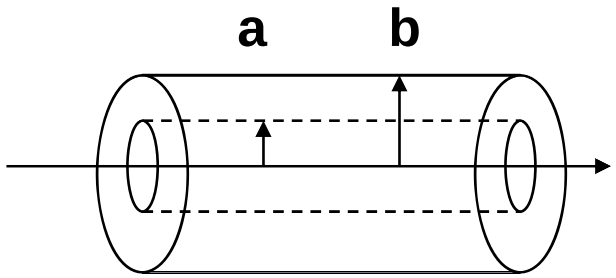
$$-j\beta E_r = -j\omega\mu H_\theta \,, \qquad -j\beta H_r = j\omega\varepsilon E_\theta$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) - \frac{1}{r}\frac{\partial E_r}{\partial \theta} = 0 \,, \qquad \frac{1}{r}\frac{\partial}{\partial r}(rH_\theta) - \frac{1}{r}\frac{\partial H_r}{\partial \theta} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\frac{\partial E_\theta}{\partial \theta} = 0 \,, \qquad \frac{1}{r}\frac{\partial}{\partial r}(rH_r) + \frac{1}{r}\frac{\partial H_\theta}{\partial \theta} = 0$$

$$\beta^2 = \omega^2 \varepsilon \mu \,, \quad k_c = 0 \,, \qquad \text{No Cut-off}$$

$$E_\theta = 0\,,\quad H_r = 0\,,$$



$$\frac{\partial}{\partial r}\left(r\,E_r\right) = 0\,,\quad \frac{\partial}{\partial r}\left(r\,H_\theta\right) = 0$$

$$E_r = \frac{\mathrm{A}}{r}e^{-j\beta z}\,,\quad H_\theta = \frac{1}{Z_i}\frac{\mathrm{A}}{r}e^{-j\beta z}$$

$$V = \int_a^b E_r\,dr = \mathrm{A}\,\mathrm{In}\frac{b}{a}e^{-j\beta z}\,,\quad I = \int_0^{2\pi} J_S\,a\,d\theta = 2\pi\frac{\mathrm{A}}{Z_i}e^{-j\beta z}$$

$$Z_0 = \frac{V}{I} = \frac{Z_i}{2\pi}\mathrm{In}\frac{b}{a}\,;$$ Characteristic Impedance

Power

$$P_f = \int_0^{2\pi} \int_a^b \frac{1}{2} \mathrm{Re}\left(E_r H_\theta^*\right) r dr d\theta = \frac{\mathrm{A}^2}{Z_i} \pi \ln \frac{b}{a}$$

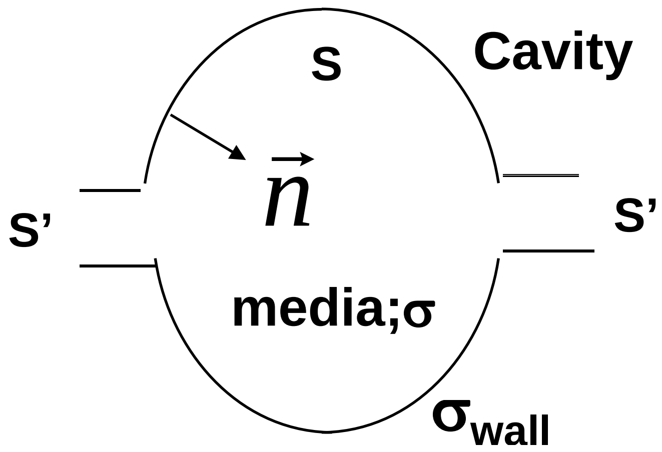
$$P_{loss} = \frac{1}{2} R_S \left[\int_0^{2\pi} \left| H_\theta \right|_{r=a}^2 a d\theta + \int_0^{2\pi} \left| H_\theta \right|_{r=b}^2 b d\theta \right] = R_S \frac{\mathrm{A}^2}{Z_i^2} \pi \left(\frac{1}{a} + \frac{1}{b} \right)$$

$$= R_S \frac{P_f}{Z_i} \left(\frac{1}{a} + \frac{1}{b} \right) \Big/ \ln \frac{b}{a} = \frac{R_S P_f}{2\pi Z_0} \left(\frac{1}{a} + \frac{1}{b} \right)$$

Resonator / Cavity

Can be solved Analytically or by Computer Codes

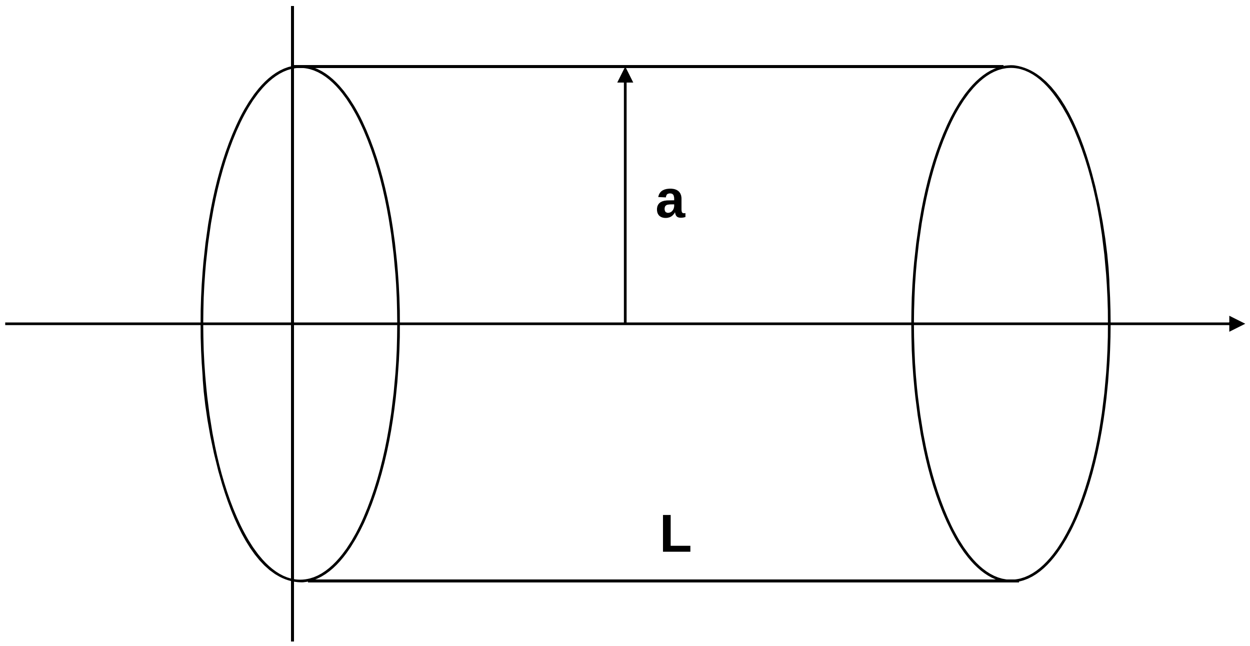
- Boundary Condition
- Short-Circuited Plane S
- Open-Circuited Plane S'



$$\vec{n} \times \vec{E}_a = 0\,,\ \ \vec{n} \cdot \vec{H}_a = 0 \ \text{ on S}$$; **Perfect Conductor**

$$\vec{n} \times \vec{H}_a = 0\,,\ \ \vec{n} \cdot \vec{E}_a = 0 \ \text{ on S'}$$

Analytic Solution, Example



$$\sin\beta z = 0\,; \quad \text{at } z = L \quad \Rightarrow \quad \beta = \frac{l\pi}{L}$$

TM-01*l* Modes

$$E_r = -j\frac{\beta}{(y_{mn}/a)}\cos m\theta J'_m\left(\frac{y_{mn}}{a}r\right)e^{-j\beta z},$$

$$E_\theta = j\frac{\beta m}{(y_{mn}/a)^2}\sin m\theta\frac{1}{r}J_m\left(\frac{y_{mn}}{a}r\right)e^{-j\beta z},$$

$$E_z = \cos m\theta\, J_m\left(\frac{y_{mn}}{a}r\right)e^{-j\beta z},$$

$$H_r = -j\frac{\omega\varepsilon m}{(y_{mn}/a)^2}\sin m\theta\frac{1}{r}J_m\left(\frac{y_{mn}}{a}r\right)e^{-j\beta z},$$

$$H_\theta = -j\frac{\omega\varepsilon}{(y_{mn}/a)}\cos m\theta J'_m\left(\frac{y_{mn}}{a}r\right)e^{-j\beta z},$$

$$H_z = 0$$

$$E_r = -j\frac{\beta}{(y_{01}/a)}J'_0\left(\frac{y_{01}}{a}r\right)\sin\frac{l\pi}{L}z,$$

$$E_\theta = 0,$$

$$E_z = J_0\left(\frac{y_{01}}{a}r\right)\cos\frac{l\pi}{L}z,$$

$$H_r = 0,$$

$$H_\theta = -j\frac{\omega\varepsilon}{(y_{01}/a)}J'_m\left(\frac{y_{mn}}{a}r\right)\cos\frac{l\pi}{L}z,$$

$$H_z = 0$$

$$\lambda_g = \frac{2\pi}{\beta} = \frac{2L}{l}, \quad \lambda_c = \frac{2\pi a}{y_{01}},$$

$$\frac{1}{\lambda} = \sqrt{\frac{1}{\lambda_g^2} + \frac{1}{\lambda_c^2}} = \sqrt{\left(\frac{l}{2L}\right)^2 + \left(\frac{y_{01}}{2\pi a}\right)^2},$$

$$f = \frac{c}{\lambda}$$

Cavity RF Parameters

$$E_{acc}=\frac{1}{L_{Cavity}}\int_0^L E_z(z,r=0)\cos\{\omega t(z)\}dz$$

$$P_0=\frac{R_S}{2}\int\left|\vec{H}\right|^2 dS,\quad W=\frac{\mu}{2}\int\left|\vec{H}\right|^2 dV=\frac{\varepsilon}{2}\int\left|\vec{E}\right|^2 dV$$

$$Q_0\equiv\frac{\omega W}{P_0}=\frac{G}{R_S},\quad G=\omega\mu\frac{\int\left|\vec{H}\right|^2 dV}{\int\left|\vec{H}\right|^2 dS}$$

Geometric Factor

$$R\equiv\frac{E_{acc}^2}{P_0}L_{Cavity}^2,\quad \left(\frac{R}{Q}\right)=\frac{E_{acc}^2}{\omega W}L_{Cavity}^2$$

$$\mathrm{TTF} = \frac{\int_0^L E_z(z, r=0)\cos\{\omega t(z)\}dz}{\int_0^L E_z(z, r=0)dz} = 0.7181$$

$$Q = \frac{a}{\delta(1 + a/L)}$$